

Cell algebra structure on generalized Schur algebras

Robert May

Department of Mathematics and Computer Science
Longwood University
201 High Street, Farmville, VA 23909
rmay@longwood.edu

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1 Introduction

A family of “generalized Schur algebras” were first introduced in [6] and [2]. In [3] the left and right generalized Schur algebras were shown to be “double coset algebras”. In [2] and [5] a stratification of these algebras was given leading to a parameterization, in most cases, of their irreducible representations. In this paper we obtain cell algebra structures for these algebras in the sense of [4]. (The Cell algebras of [4] coincide with the standardly based algebras previously introduced by Du and Rui in [8].) The properties of cell algebras combined with the parameterization of the irreducible representations leads to a more concrete description of all these irreducibles. In certain cases these algebras are shown to be quasi-hereditary.

In section 2 we review the definition and properties of cell algebras as presented in [4]. In section 3 we describe the cell bases found in [4] for a family of semigroups including the full transformation semigroups \mathcal{T}_r and the rook semigroups \mathfrak{R}_r . In section 4 we give cell bases for the left and right generalized Schur algebras corresponding to these semigroups. Finally, in section 5 we use the cell algebra structure to describe the irreducible representations of these algebras and to determine when they are quasi-hereditary.

2 Cell algebra structures

In this section we review, without proofs, the definition and properties of cell algebras as presented in [4]. (These algebras were previously studied as “standardly based algebras” by Du and Rui in [8].) Let R be a commutative integral domain with unit 1 and let A be an associative, unital R -algebra. Let Λ be a finite set with a partial order \leq and for each $\lambda \in \Lambda$ let $L(\lambda), R(\lambda)$

be finite sets of “left indices” and “right indices”. Assume that for each $\lambda \in \Lambda$, $s \in L(\lambda)$, and $t \in R(\lambda)$ there is an element ${}_s C_t^\lambda \in A$ such that the map $(\lambda, s, t) \mapsto {}_s C_t^\lambda$ is injective and

$$C = \{{}_s C_t^\lambda : \lambda \in \Lambda, s \in L(\lambda), t \in R(\lambda)\}$$

is a free R -basis for A . Define R -submodules of A by

$$A^\lambda = R\text{-span of } \{{}_s C_t^\mu : \mu \in \Lambda, \mu \geq \lambda, s \in L(\mu), t \in R(\mu)\}$$

and

$$\hat{A}^\lambda = R\text{-span of } \{{}_s C_t^\mu : \mu \in \Lambda, \mu > \lambda, s \in L(\mu), t \in R(\mu)\}.$$

Definition 2.1. Given A, Λ, C , A is a cell algebra with poset Λ and cell basis C if

- i For any $a \in A, \lambda \in \Lambda$, and $s, s' \in L(\lambda)$, there exists $r_L = r_L(a, \lambda, s, s') \in R$ such that, for any $t \in R(\lambda)$, $a \cdot {}_s C_t^\lambda = \sum_{s' \in L(\lambda)} r_L \cdot {}_{s'} C_t^\lambda \pmod{\hat{A}^\lambda}$, and
- ii For any $a \in A, \lambda \in \Lambda$, and $t, t' \in R(\lambda)$, there exists $r_R = r_R(a, \lambda, t, t') \in R$ such that, for any $s \in L(\lambda)$, ${}_s C_t^\lambda \cdot a = \sum_{t' \in R(\lambda)} r_R \cdot {}_s C_{t'}^\lambda \pmod{\hat{A}^\lambda}$.

Consider a fixed cell algebra A with poset Λ and cell basis C .

Lemma 2.1.

- (a) A^λ and \hat{A}^λ are two sided ideals in A for any $\lambda \in \Lambda$.
- (b) For $\lambda \in \Lambda, t, t' \in R(\lambda), s, s' \in L(\lambda)$, $r_L({}_{s'} C_t^\lambda, \lambda, s, s') = r_R({}_s C_{t'}^\lambda, \lambda, t, t')$.
- (c) Given $\lambda \in \Lambda, t \in R(\lambda), s \in L(\lambda)$, there exists $r_{st} \in R$ such that for any $s' \in L(\lambda), t' \in R(\lambda)$ we have ${}_{s'} C_t^\lambda {}_s C_{t'}^\lambda = r_{st} {}_{s'} C_{t'}^\lambda \pmod{\hat{A}^\lambda}$. In fact $r_{st} = r_L({}_{s'} C_t^\lambda, \lambda, s, s') = r_R({}_s C_{t'}^\lambda, \lambda, t, t')$.

By lemma 2.1, part (a), A/\hat{A}^λ is a unital R -algebra and $A^\lambda/\hat{A}^\lambda$ is a two sided ideal in A/\hat{A}^λ . Observe that as an R -module $A^\lambda/\hat{A}^\lambda$ is free with a basis $\{{}_s C_t^\lambda + \hat{A}^\lambda : s \in L(\lambda), t \in R(\lambda)\}$.

For a fixed $t \in R(\lambda)$, define ${}_L C_t^\lambda$ as the free R -submodule of $A^\lambda/\hat{A}^\lambda$ with basis $\{{}_s C_t^\lambda + \hat{A}^\lambda : s \in L(\lambda)\}$. By property (i), ${}_L C_t^\lambda$ is a left A -module and ${}_L C_t^\lambda \cong {}_L C_{t'}^\lambda$ as left A -modules for any $t, t' \in R(\lambda)$. Evidently, as left A -modules we have $A^\lambda/\hat{A}^\lambda \cong \bigoplus_{t \in R(\lambda)} {}_L C_t^\lambda$.

Definition 2.2. The left cell module for λ is the left A -module ${}_L C^\lambda$ defined as follows: Take the free R -module with a basis $\{{}_s C^\lambda : s \in L(\lambda)\}$ and define the left action of A by $a \cdot {}_s C^\lambda = \sum_{s' \in L(\lambda)} r_L(a, \lambda, s, s') {}_{s'} C^\lambda$ for $a \in A$.

For any $t \in R(\lambda)$, ${}_s C^\lambda_t \mapsto {}_s C^\lambda_t + \hat{A}^\lambda$ gives a left A -module isomorphism $\phi_t : {}_L C^\lambda \rightarrow {}_L C^\lambda_t$. Then $A^\lambda/\hat{A}^\lambda \cong \bigoplus_{t \in R(\lambda)} {}_L C^\lambda_t$ is isomorphic to the direct sum of $|R(\lambda)|$ copies of ${}_L C^\lambda$.

In a parallel way, for a fixed $s \in L(\lambda)$, define ${}_s C^\lambda_R$ as the free R -module with basis $\{{}_s C^\lambda_t + \hat{A}^\lambda : t \in R(\lambda)\}$, an R -submodule of $A^\lambda/\hat{A}^\lambda$. By property (ii), ${}_s C^\lambda_R$ is a right A -module and ${}_s C^\lambda_R \cong {}_{s'} C^\lambda_R$ as right A -modules for any $s, s' \in L(\lambda)$. As right A -modules we have $A^\lambda/\hat{A}^\lambda \cong \bigoplus_{s \in L(\lambda)} {}_s C^\lambda_R$.

Definition 2.3. *The right cell module for λ is the right A -module C^λ_R defined as follows: Take the free R -module with a basis $\{C^\lambda_t : t \in R(\lambda)\}$ and define the right action of A by $C^\lambda_t \cdot a = \sum_{t' \in R(\lambda)} r_{st'}(a, \lambda, t, t') C^\lambda_{t'}$ for $a \in A$.*

For any $s \in L(\lambda)$, $C^\lambda_t \mapsto {}_s C^\lambda_t + \hat{A}^\lambda$ gives a right A -module isomorphism ${}_s \phi : C^\lambda_R \rightarrow {}_s C^\lambda_R$. Then $A^\lambda/\hat{A}^\lambda \cong \bigoplus_{s \in L(\lambda)} {}_s C^\lambda_R$ is isomorphic to the direct sum of $|L(\lambda)|$ copies of C^λ_R .

For each $\lambda \in \Lambda$ there is an R -bilinear map $\langle \cdot, \cdot \rangle : C^\lambda_R \times {}_L C^\lambda \rightarrow R$ defined on basis elements by $\langle C^\lambda_t, {}_s C^\lambda \rangle = r_{st}$, where $r_{st} \in R$ is as given in lemma 2.1.

Definition 2.4. *The right C^λ_R radical is*

$$\text{rad}(C^\lambda_R) = \{x \in C^\lambda_R : \langle x, y \rangle = 0 \text{ for all } y \in {}_L C^\lambda\}.$$

The left ${}_L C^\lambda$ radical is

$$\text{rad}({}_L C^\lambda) = \{y \in {}_L C^\lambda : \langle x, y \rangle = 0 \text{ for all } x \in C^\lambda_R\}.$$

The radical $\text{rad}(C^\lambda_R)$ is a right A -submodule of C^λ_R and $\text{rad}({}_L C^\lambda)$ is a left A -submodule of ${}_L C^\lambda$.

Definition 2.5. $D^\lambda_R = \frac{C^\lambda_R}{\text{rad}(C^\lambda_R)}$, ${}_L D^\lambda = \frac{{}_L C^\lambda}{\text{rad}({}_L C^\lambda)}$.

Then D^λ_R is a right A -module and ${}_L D^\lambda$ is a left A -module. The following lemma follows at once from the definitions.

Lemma 2.2. *The following conditions are equivalent:*

- (i) $D^\lambda_R = 0$; (ii) $\text{rad}(C^\lambda_R) = C^\lambda_R$; (iii) $\langle x, y \rangle = 0$ for all $x \in C^\lambda_R, y \in {}_L C^\lambda$;
- (iv) $\text{rad}({}_L C^\lambda) = {}_L C^\lambda$; and (v) ${}_L D^\lambda = 0$.

Definition 2.6. $\Lambda_0 = \{\lambda \in \Lambda : \langle x, y \rangle \neq 0 \text{ for some } x \in C^\lambda_R, y \in {}_L C^\lambda\}$.

Evidently, $\lambda \in \Lambda_0 \Leftrightarrow D^\lambda_R \neq 0 \Leftrightarrow {}_L D^\lambda \neq 0$.

When $R = k$ is a field, one can characterize the irreducible modules in a cell algebra in terms of the set Λ_0 .

Proposition 2.1. *Let $R = k$ be a field and take $\lambda \in \Lambda_0$. Then*

- (a) D_R^λ is an irreducible right A -module.
- (b) $\text{rad}(C_R^\lambda)$ is the unique maximal right submodule in C_R^λ .
- (c) ${}_L D^\lambda$ is an irreducible left A -module.
- (d) $\text{rad}({}_L C^\lambda)$ is the unique maximal left submodule in ${}_L C^\lambda$.

The modules $\{D_R^\lambda : \lambda \in \Lambda_0\}$ are shown to be absolutely irreducible and pairwise inequivalent and similarly for $\{{}_L D^\lambda : \lambda \in \Lambda_0\}$.

A major result of [4] is the following:

Theorem 2.1. *Assume $R = k$ is a field. Then $\{D_R^\mu : \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible right A -modules and $\{{}_L D^\mu : \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible left A -modules.*

In [4] it the following result is also obtained:

Corollary 2.1. *If $\Lambda = \Lambda_0$, then A is quasi-hereditary.*

3 Cell bases for certain monoid algebras

In this section we review (again omitting most of the proofs) the cell bases given in [4] for the monoid algebras $R[M]$ corresponding to a class of monoids M containing the full transformation semigroups \mathcal{T}_r and the rook monoids \mathfrak{R}_r . Some of the notation and results will be needed in the next section on generalized Schur algebras.

Let $\bar{r} = \{1, 2, \dots, r\}$ and let $\bar{\tau}_r$ be the monoid of all maps $\alpha : \bar{r} \cup \{0\} \rightarrow \bar{r} \cup \{0\}$ such that $\alpha(0) = 0$. Note that $\bar{\tau}_r$ can be identified with the partial transformation semigroup \mathcal{PT}_r of all “partial maps” of \bar{r} to itself. The full transformation semigroup \mathcal{T}_r of all maps $\bar{r} \rightarrow \bar{r}$ can be identified with the submonoid of $\bar{\tau}_r$ consisting of maps with $\alpha^{-1}(0) = 0$. The rook monoid \mathfrak{R}_r can be identified with the submonoid of $\bar{\tau}_r$ consisting of maps such that $\alpha^{-1}(i)$ has at most one element for each $i \in \bar{r}$. With these identifications, the symmetric group \mathfrak{S}_r is the intersection $\mathcal{T}_r \cap \mathfrak{R}_r$.

Let M be any monoid contained in $\bar{\tau}_r$ and containing \mathfrak{S}_r . Let R be a commutative domain with unit 1 and let $R[M]$ be the monoid algebra over R . We will describe a cell basis for $R[M]$.

For $\alpha \in M$, the index of α is the number of nonzero elements in the image of α , $\text{index}(\alpha) = |\text{image}(\alpha) - \{0\}|$. Let $I(M) \subseteq \bar{r} \cup \{0\}$ be the set of indices of elements in M , that is,

$$I(M) = \{i : \exists \alpha \in M \text{ with } \text{index}(\alpha) = i\}.$$

For $i \in \bar{r}$, let $\Lambda(i)$ be the set of all (integer) partitions of i . Let $\Lambda(0)$ be a set with one element λ_0 . Then define $\Lambda = \cup_{i \in I(M)} \Lambda(i)$. For $\lambda \in \Lambda$ define the index $i(\lambda)$ to be the integer such that $\lambda \in \Lambda(i(\lambda))$. Finally, define a partial order on Λ by

$$\lambda \geq \mu \Leftrightarrow i(\lambda) < i(\mu) \text{ or } i(\lambda) = i(\mu) \text{ and } \lambda \succeq \mu$$

where \triangleright is the usual dominance relation on partitions.

To define the sets $L(\lambda)$ and $R(\lambda)$ we need some preliminaries. First, for $i \in \bar{r} \cup \{0\}$ let $C(i, r)$ be the collection of all i -sets of elements in \bar{r} , that is, $C(i, r) = \{C = \{c_1, c_2, \dots, c_i\} : 1 \leq c_1 < c_2 < \dots < c_i \leq r\}$. ($C(0, r)$ contains one element, the empty set.) For any $C \in C(i, r)$, define a map $\phi_C : \bar{i} \cup \{0\} \rightarrow \bar{r} \cup \{0\}$ by $\phi_C(j) = c_j$ for $j \in \bar{i}$, $\phi_C(0) = 0$.

Next, choose any ordering of the 2^r subsets of \bar{r} and label these subsets d_j so that $d_1 < d_2 < \dots < d_{2^r}$. Let $D(i, r)$ be the collection of sets of i nonempty, pairwise disjoint subsets of \bar{r} , that is,

$$D(i, r) = \{\{d_{a_1}, d_{a_2}, \dots, d_{a_i}\} : d_{a_j} \neq \emptyset, d_{a_j} \cap d_{a_k} = \emptyset \text{ and } d_{a_j} < d_{a_k} \text{ for } j < k\}.$$

($D(0, r)$ also contains one element, the empty set.) For any $D \in D(i, r)$ define a map $\psi_D : \bar{r} \cup \{0\} \rightarrow \bar{i} \cup \{0\}$ by $\psi_D(x) = j$ for $x \in d_{a_j}$, $\psi_D(x) = 0$ when $x \notin d_{a_j}$ for any j .

Regard an element σ in the symmetric group \mathfrak{S}_i as a mapping $\sigma : \bar{i} \cup \{0\} \rightarrow \bar{i} \cup \{0\}$ such that $\sigma(0) = 0$. Then for any $\sigma \in \mathfrak{S}_i$, $C \in C(i, r)$, $D \in D(i, r)$, define an element $\alpha = \alpha(\sigma, C, D) \in \bar{r}$ by $\alpha = \phi_C \circ \sigma \circ \psi_D$. Then $\alpha(\sigma, C, D)$ has index i . Note that $\alpha(x) = \begin{cases} c_{\sigma(j)} & \text{if } x \in D_{\alpha_j} \\ 0 & \text{if } x \notin D_{\alpha_j} \text{ for any } j \end{cases}$.

Lemma 3.1. *For any $\alpha \in \bar{r}$ of index i , there exist unique $\sigma_\alpha \in \mathfrak{S}_i$, $C_\alpha \in C(i, r)$, $D_\alpha \in D(i, r)$ such that $\alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha}$.*

Notice that \mathfrak{S}_r acts on the left on $C(i, r)$: for $C = \{c_1, c_2, \dots, c_i\} \in C(i, r)$ and $\sigma \in \mathfrak{S}_r$, let $\sigma C = \{\sigma c_1, \sigma c_2, \dots, \sigma c_i\}$.

Lemma 3.2. *Given $C, C' \in C(i, r)$ and $\pi \in \mathfrak{S}_i$, there exists a $\sigma \in \mathfrak{S}_r$ such that $C' = \sigma C$ and $\sigma \circ \phi_C = \phi_{C'} \circ \pi$.*

Given $C \in C(i, r)$, $D \in D(i, r)$, let $A(C, D)$ be the free R -submodule of $R[\bar{r}]$ with basis $\{\alpha \in \bar{r} : C_\alpha = C, D_\alpha = D\}$. Note that if $i = 0$, then $C(0, r) = D(0, r) = \{\emptyset\}$, a set with one element. $A(\emptyset, \emptyset)$ is then one dimensional with basis z , where z is the zero map such that $z(j) = 0$ for all $j \in \bar{r} \cup \{0\}$. Evidently, as an R -module

$$R[\bar{r}] = \bigoplus_{i \in \bar{r} \cup \{0\}} \left(\bigoplus_{C \in C(i, r), D \in D(i, r)} A(C, D) \right).$$

Lemma 3.3. *Suppose that for $D \in D(i, r)$ there exists an $\alpha \in M$ with $D_\alpha = D$. Then $A(C, D) \cap R[M] = A(C, D)$ for every $C \in C(i, r)$.*

Define $D(M, i, r) = \{D \in D(i, r) : \exists \alpha \in M \text{ with } D_\alpha = D\}$. Then as an R -module, $R[M] = \bigoplus_{i \in \bar{r} \cup \{0\}} \left(\bigoplus_{C \in C(i, r), D \in D(M, i, r)} A(C, D) \right)$ by lemma 3.3. So choosing a basis for each free R -module $A(C, D)$, $C \in C(i, r)$, $D \in D(M, i, r)$ will give a basis for $R[M]$.

Definition 3.1. *For $C \in C(i, r)$, $D \in D(i, r)$, $i > 0$, define a map of R -modules $H_{C, D} : R[\mathfrak{S}_i] \rightarrow A(C, D)$ by $H_{C, D}(\sigma) = \phi_C \circ \sigma \circ \psi_D$.*

By lemma 3.1, $H_{C,D}$ is well-defined and is a bijection between free R -modules. So any basis for $R[\mathfrak{S}_i]$ transfers to a basis for $A(C,D)$. Let $B_i = \{ {}_s C_t^\lambda : \lambda \in \Lambda(i), s, t \text{ standard } \lambda \text{ tableaux} \}$ be the standard Murphy cellular basis for the cellular algebra $R[\mathfrak{S}_i]$ (See e.g. [1] or [7]). Then $\{ H_{C,D}({}_s C_t^\lambda) : {}_s C_t^\lambda \in B_i \}$ is a basis for $A(C,D)$.

We can now finally define our index sets $L(\lambda)$ and $R(\lambda)$. Given $\lambda \in \Lambda(i)$, $i \in I(M)$, $i > 0$, define

$$L(\lambda) = \{ (C, s) : C \in C(i, r), s \text{ a standard } \lambda \text{ tableau} \}$$

and

$$R(\lambda) = \{ (D, t) : D \in D(M, i, r), t \text{ a standard } \lambda \text{ tableau} \}.$$

Then for any $\lambda \in \Lambda$, $(C, s) \in L(\lambda)$, $(D, t) \in R(\lambda)$ define

$$({}_{(C,s)} C_{(D,t)}^\lambda = H_{C,D}({}_s C_t^\lambda) \in A(C, D) \subseteq R[M].$$

If $0 \in I(M)$, that is, if the zero map z such that $z(j) = 0$ for all $j \in \bar{r} \cup \{0\}$ is in M , we define $\Lambda(0)$ to have a single element λ_0 and define $L(\lambda_0) = R(\lambda_0) = \{\emptyset\}$ each to be sets containing one element, \emptyset . Then define ${}_\emptyset C_\emptyset^{\lambda_0} = z$. The set

$$\left\{ ({}_{(C,s)} C_{(D,t)}^\lambda) : \lambda \in \Lambda, (C, s) \in L(\lambda), (D, t) \in R(\lambda) \right\}$$

is a union of the bases for the various direct summands $A(C, D)$ and is therefore a basis for the free R -module $R[M]$. We will show that it is a cell-basis for $R[M]$.

Write A_i for the cellular algebra $R[\mathfrak{S}_i]$ and \hat{A}_i^λ for the two sided ideal in $R[\mathfrak{S}_i]$ spanned by $\{ {}_s C_t^\mu : \mu > \lambda \}$. The following observation will be useful. Recall that \hat{A}^λ is the R -submodule of $R[M]$ spanned by $\{ ({}_{(C,s)} C_{(D,t)}^\mu) : \mu > \lambda \}$.

Lemma 3.4. *For any $C \in C(i, r)$, $D \in D(i, r)$ and $\lambda \in \Lambda(i)$, $H_{C,D}(\hat{A}_i^\lambda) \subseteq \hat{A}^\lambda$.*

Lemma 3.5. *For $\alpha \in M$, $C \in C(i, r)$, suppose $C' = \alpha(C) \in C(i, r)$. Then there exists $\rho \in \mathfrak{S}_i$ such that $\alpha \circ \phi_C = \phi_{C'} \circ \rho$.*

Lemma 3.6. *For $\alpha \in M$, $D = \{ d_{a_j} : j \in \bar{i} \} \in D(i, r)$, suppose that $\alpha^{-1}(d_{a_j}) \neq \emptyset$ for all j , so that $D' = \{ \alpha^{-1}(d_{a_j}) : j \in \bar{i} \} \in D(i, r)$. Then there exists $\rho \in \mathfrak{S}_i$ such that $\psi_D \circ \alpha = \rho \circ \psi_{D'}$. Furthermore, if $D \in D(M, i, r)$, then $D' \in D(M, i, r)$.*

Proposition 3.1. *$C = \{ ({}_{(C,s)} C_{(D,t)}^\lambda) : \lambda \in \Lambda, (C, s) \in L(\lambda), (D, t) \in R(\lambda) \}$ is a cell basis for $A = R[M]$.*

For $\lambda \in \Lambda$, the right cell module C_R^λ is a right A -module and a free R -module with basis $\{ C_{(D,t)}^\lambda : (D, t) \in R(\lambda) \}$, while the left cell module ${}_L C^\lambda$ is a left A -module and a free R -module with basis $\{ ({}_{(C,s)} C^\lambda) : (C, s) \in L(\lambda) \}$. The

bracket for A is an R -bilinear map $\langle -, - \rangle : C_R^\lambda \times {}_L C^\lambda \rightarrow R$ defined on basis elements by $\langle C_{(D,t)}^\lambda, {}_{(C,s)} C^\lambda \rangle = r_{(C,s),(D,t)} \in R$ where ${}_{(C',s')} C_{(D,t)}^\lambda \cdot {}_{(C,s)} C_{(D',t')}^\lambda = r_{(C,s),(D,t)} \cdot {}_{(C',s')} C_{(D',t')}^\lambda \pmod{\hat{A}^\lambda}$.

Lemma 3.7. *Assume $C, C' \in C(i, r)$, $D, D' \in D(i, r)$ and $x, y \in R(\mathfrak{S}_i)$. Then*

- (a) *If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is not bijective, then $H_{C',D}(x) \cdot H_{C,D'}(y) \in J_{i-1}$.*
- (b) *If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is bijective, then for any $\pi \in \mathfrak{S}_i$, $H_{C',D}(x) \cdot H_{C,D'}(\pi y) = H_{C',D'}(x\rho\pi y)$.*

Lemma 3.8. *Let $i \in I(M)$, $\lambda \in \Lambda(i)$, $(C, s) \in L(\lambda)$, and $(D, t) \in R(\lambda)$. Then*

- (a) *If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is not bijective, $\langle C_{(D,t)}^\lambda, {}_{(C,s)} C^\lambda \rangle = 0$.*
- (b) *If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is bijective, then for any $\pi \in \mathfrak{S}_i$,*

$$\langle C_{(D,t)}^\lambda, \pi' \cdot {}_{(C,s)} C^\lambda \rangle = \langle C_t^\lambda, \rho\pi \cdot {}_s C^\lambda \rangle_i.$$

Here $\langle -, - \rangle_i$ is the bracket in the cellular algebra $R[\mathfrak{S}_i]$ and $\pi' = H_{C,D''}(\pi)$ for any $D'' \in D(M, i, r)$ such that $\psi_{D''} \circ \phi_C = \text{id} : \bar{i} \rightarrow \bar{i}$.

Recall that the radical, $\text{rad}(C_R^\lambda)$, of a right cell module is the right A -module given by $\text{rad}(C_R^\lambda) = \{x \in C_R^\lambda : \langle x, y \rangle = 0 \text{ for all } y \in {}_L C^\lambda\}$.

Proposition 3.2. *Let $i \in I(M)$, $\lambda \in \Lambda(i)$. Then $\text{rad}(C_R^\lambda) = C_R^\lambda$ in $A \Leftrightarrow \text{rad}(C^\lambda) = C^\lambda$ in A_i .*

Proof. Assume first that $\text{rad}(C^\lambda) = C^\lambda$ in A_i , so $\langle x, y \rangle_i = 0$ for all x, y . To show $\text{rad}(C_R^\lambda) = C_R^\lambda$ it suffices to show that $\langle C_{(D,t)}^\lambda, {}_{(C,s)} C^\lambda \rangle = 0$ for any $(D, t) \in R(\lambda), (C, s) \in L(\lambda)$. If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is not bijective, lemma 3.8(a) gives $\langle C_{(D,t)}^\lambda, {}_{(C,s)} C^\lambda \rangle = 0$. If $\rho = \psi_D \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is bijective, take π in lemma 3.8 to be the identity so that $\pi' {}_{(C,s)} C^\lambda = {}_{(C,s)} C^\lambda$. Then by lemma 3.8(b), $\langle C_{(D,t)}^\lambda, {}_{(C,s)} C^\lambda \rangle = \langle C_{(D,t)}^\lambda, \pi' \cdot {}_{(C,s)} C^\lambda \rangle = \langle C_t^\lambda, \rho \cdot {}_s C^\lambda \rangle_i = 0$.

Now assume $\text{rad}(C_R^\lambda) = C_R^\lambda$ in A , so $\langle x, y \rangle = 0$ for any $x \in C_R^\lambda, y \in {}_L C^\lambda$. To show $\text{rad}(C^\lambda) = C^\lambda$ in A_i it suffices to show that $\langle C_t^\lambda, {}_s C^\lambda \rangle_i = 0$ for any t, s . Take any $D \in D(M, i, r)$ and choose $C \in C(i, r)$ such that $\rho = \psi_D \circ \phi_C$ is bijective. Then apply lemma 3.8(b) with $\pi = \rho^{-1}$ to get $\langle C_t^\lambda, {}_s C^\lambda \rangle_i = \langle C_t^\lambda, \rho\pi \cdot {}_s C^\lambda \rangle_i = \langle C_{(D,t)}^\lambda, \pi' \cdot {}_{(C,s)} C^\lambda \rangle = 0$. \square

Note: In the special case when $0 \in I(M)$, so $\Lambda(0) = \{\lambda_0\} \subseteq \Lambda$, the cell modules $C_R^{\lambda_0}, {}_L C^{\lambda_0}$ are one dimensional with generators $C_\emptyset^{\lambda_0}, {}_\emptyset C^{\lambda_0}$ and $\langle C_\emptyset^{\lambda_0}, {}_\emptyset C^\lambda \rangle = 1$ (since $z \cdot z = z$ where $z = {}_\emptyset C_\emptyset^{\lambda_0} : j \mapsto 0$ for all $j \in \bar{i} \cup \{0\}$). Then $\text{rad}(C_R^{\lambda_0}) = 0$.

4 Generalized Schur algebras

For a monoid M as in section 3 and a domain R , a “generalized Schur algebra”, $S(M, R)$, was defined in [6] and [2]. This algebra is isomorphic to $R \otimes A^{\mathbb{Z}}$ where $A^{\mathbb{Z}}$ is a certain “ \mathbb{Z} -form” . As shown in [3], there are actually two relevant \mathbb{Z} -forms, the left and right generalized Schur algebras $LGS(M, \mathbf{G}) = A_L^{\mathbb{Z}}$ and $RGS(M, \mathbf{G}) = A_R^{\mathbb{Z}}$, corresponding to the monoid M and the family of subgroups $\mathbf{G} = \{\mathfrak{S}_{\mu} : \mu \in \Lambda(r, n)\}$. Here $\Lambda(r, n)$ is the set of all compositions of r with n parts, \mathfrak{S}_{μ} is the “Young subgroup” corresponding to $\mu \in \Lambda(r, n)$, and we will always assume $n \geq r$. We sketch the description of these two algebras; for details see [3].

For compositions $\mu, \nu \in \Lambda(r, n)$, let ${}^{\mu}A^{\nu}$ be the \mathbb{Z} -submodule of $A = \mathbb{Z}[M]$ which is invariant under the action of \mathfrak{S}_{μ} on the left and \mathfrak{S}_{ν} on the right. Let ${}_{\mu}M_{\nu}$ be the set of double cosets of the form $\mathfrak{S}_{\mu}m\mathfrak{S}_{\nu}$, $m \in M$. For $\mathbf{D} \in {}_{\mu}M_{\nu}$, define $X(\mathbf{D}) = \sum_{m \in \mathbf{D}} m \in \mathbb{Z}[M]$. Then ${}^{\mu}A^{\nu}$ is a free \mathbb{Z} -module with basis $\{X(\mathbf{D}) : \mathbf{D} \in {}_{\mu}M_{\nu}\}$. Let $\bar{A} = \bigoplus_{\mu, \nu \in \Lambda(r, n)} {}^{\mu}A^{\nu}$, the direct sum of disjoint copies of submodules of A . Then \bar{A} is a free \mathbb{Z} module with basis $\{b_{\mathbf{D}} = X(\mathbf{D}) : \mathbf{D} \in {}_{\mu}M_{\lambda}, \mu, \nu \in \Lambda(r, n)\}$. Notice that if $D_1 \in {}_{\mu}M_{\nu}$, $D_2 \in {}_{\nu}M_{\pi}$, then the product $X(D_1)X(D_2)$ (defined in A) is invariant under multiplication by \mathfrak{S}_{μ} on the left and by \mathfrak{S}_{π} on the right, i.e., $X(D_1)X(D_2) \in {}^{\mu}A^{\pi}$. It is therefore a \mathbb{Z} -linear combination of $\{X(D) : D \in {}_{\mu}M_{\pi}\}$: $X(D_1)X(D_2) = \sum_{D \in {}_{\mu}M_{\pi}} a(D_1, D_2, D)X(D)$, with coefficients $a(D_1, D_2, D) \in \mathbb{Z}$. Then an associative, bilinear product on \bar{A} is defined on the basis elements $b_{D_i} = X(D_i)$ corresponding to $D_1 \in {}_{\mu}M_{\nu}$, $D_2 \in {}_{\rho}M_{\pi}$ by

$$b_{D_1}b_{D_2} = \begin{cases} \sum_{D \in {}_{\mu}M_{\pi}} a(D_1, D_2, D)b_D & \text{if } \nu = \rho \\ 0 & \text{if } \nu \neq \rho. \end{cases}$$

\bar{A} with this multiplication fails (in general) to have an identity. To obtain the \mathbb{Z} -forms $A_L^{\mathbb{Z}}$ and $A_R^{\mathbb{Z}}$, which are \mathbb{Z} -algebras with identity, we define new “left” and “right” products, $*_L$ and $*_R$ on \bar{A} .

For $\mathbf{D} \in {}_{\mu}M_{\nu}$, let $n_L(\mathbf{D})$ be the number of elements in any left \mathfrak{S}_{μ} -coset $C \subseteq \mathbf{D}$. Then the product $*_L$ is defined on the basis elements $b_{\mathbf{D}_i} = X(\mathbf{D}_i)$ corresponding to $\mathbf{D}_1 \in {}_{\mu}M_{\nu}$, $\mathbf{D}_2 \in {}_{\rho}M_{\pi}$ by

$$b_{\mathbf{D}_1} *_L b_{\mathbf{D}_2} = \begin{cases} \sum_{\mathbf{D} \in {}_{\mu}M_{\pi}} \frac{n_L(\mathbf{D})}{n_L(\mathbf{D}_1)n_L(\mathbf{D}_2)} a(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}) b_{\mathbf{D}} & \text{if } \nu = \rho \\ 0 & \text{if } \nu \neq \rho. \end{cases}$$

Similarly, for $\mathbf{D} \in {}_{\mu}M_{\nu}$, let $n_R(\mathbf{D})$ be the number of elements in any right \mathfrak{S}_{ν} -coset $C \subseteq \mathbf{D}$. Then the product $*_R$ is defined on the basis elements $b_{\mathbf{D}_i} =$

$X(\mathbf{D}_i)$ corresponding to $\mathbf{D}_1 \in {}_\mu M_\nu, \mathbf{D}_2 \in {}_\rho M_\pi$ by

$$b_{\mathbf{D}_1} *_R b_{\mathbf{D}_2} = \begin{cases} \sum_{\mathbf{D} \in {}_\mu M_\pi} \frac{n_R(\mathbf{D})}{n_R(\mathbf{D}_1)n_R(\mathbf{D}_2)} a(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}) b_{\mathbf{D}} & \text{if } \nu = \rho \\ 0 & \text{if } \nu \neq \rho. \end{cases}$$

As shown in [3], the structure constants for these multiplications are in fact in \mathbb{Z} and the resulting \mathbb{Z} -algebras, $A_L^{\mathbb{Z}}$ and $A_R^{\mathbb{Z}}$, have identities. We will show that both of these algebras, and the associated generalized Schur algebras $R \otimes A_L^{\mathbb{Z}}$ and $R \otimes A_R^{\mathbb{Z}}$ for any domain R , have cell bases and are cell algebras.

We use the notation and definitions of section 3. For a composition $\mu \in \Lambda(r, n)$, \mathfrak{S}_μ acts on the left on $C(i, r)$: if $C = \{c_1, c_2, \dots, c_i\} \in C(i, r)$ and $\sigma \in \mathfrak{S}_\mu$ define $\sigma C = \{\sigma c_1, \dots, \sigma c_i\}$. Then write $O(\mu, C) = \{\rho C : \rho \in \mathfrak{S}_\mu\}$ for the orbit of $C \in C(i, r)$ under \mathfrak{S}_μ . Similarly, for a composition $\nu \in \Lambda(r, n)$, \mathfrak{S}_ν acts on the right on $D(M, i, r)$: for $D = \{d_1, d_2, \dots, d_i\} \in D(M, i, r)$ and $\pi \in \mathfrak{S}_\nu$, $D\pi = \{\pi^{-1}[d_1], \dots, \pi^{-1}[d_i]\}$. Write $O(\nu, D) = \{D\pi : \pi \in \mathfrak{S}_\nu\}$ for the orbit of $D \in D(M, i, r)$ under \mathfrak{S}_ν . Any double coset $\mathbf{D} = \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu \in {}_\mu M_\nu$ has a well defined index i since any $\beta \in \mathbf{D}$ has the same index as α . We also have $O(\mu, C_\beta) = O(\mu, C_\alpha)$ for any $\beta \in \mathbf{D}$, so \mathbf{D} has a well defined orbit $O(\mu, \mathbf{D}) = O(\mu, C_\alpha)$ for the action of \mathfrak{S}_μ on $C(i, r)$. Similarly, \mathbf{D} has a well defined orbit $O(\nu, \mathbf{D}) = O(\nu, D_\alpha)$ for the action of \mathfrak{S}_ν on $D(M, i, r)$. Let $\mathbf{O}_{\mu, C(i, r)}$ be the set of orbits for the action of \mathfrak{S}_μ on $C(i, r)$ so $C(i, r) = \bigcup_{O \in \mathbf{O}_{\mu, C(i, r)}} O$.

Similarly, let $\mathbf{O}_{\nu, D(M, i, r)}$ be the set of orbits for the action of \mathfrak{S}_ν on $D(M, i, r)$ so $D(M, i, r) = \bigcup_{O \in \mathbf{O}_{\nu, D(M, i, r)}} O$. Define the set of orbit pairs $\mathbf{O}(\mu, \nu, M, i) = \{(O_\mu, O_\nu) : O_\mu \in \mathbf{O}_{\mu, C(i, r)}, O_\nu \in \mathbf{O}_{\nu, D(M, i, r)}\}$. Finally, for $(O_\mu, O_\nu) \in \mathbf{O}(\mu, \nu, M, i)$ define a collection of double cosets

$$M(O_\mu, O_\nu) = \{\mathbf{D} \in {}_\mu M_\nu : \text{index}(\mathbf{D}) = i, O(\mu, \mathbf{D}) = O_\mu, O(\nu, \mathbf{D}) = O_\nu\}.$$

Then for orbits $O_\mu \in \mathbf{O}_{\mu, C(i, r)}$, $O_\nu \in \mathbf{O}_{\nu, D(M, i, r)}$ define ${}^{O_\mu} A^{O_\nu}$ to be the free \mathbb{Z} -module with basis $\{X(\mathbf{D}) : \mathbf{D} \in M(O_\mu, O_\nu)\}$. Evidently

$${}^\mu A^\nu = \bigoplus_{i \in I(M)} \bigoplus_{(O_\mu, O_\nu) \in \mathbf{O}(\mu, \nu, M, i)} {}^{O_\mu} A^{O_\nu}.$$

We will obtain a cell basis for $\bar{A} = \bigoplus_{\mu, \nu \in \Lambda(r, n)} {}^\mu A^\nu$ by taking the union of bases for the individual submodules ${}^{O_\mu} A^{O_\nu}$.

For $\mu \in \Lambda(r, n)$ and $C \in C(i, r)$ let $\mu(C) \in \Lambda(i, n)$ be the composition of i obtained as follows: if b_j^μ , $j \in \bar{n}$, is the j th block of μ , then the j th block of $\mu(C)$ is given by $b_j^{\mu(C)} = \phi_C^{-1}(b_j^\mu)$, $j \in \bar{n}$. Notice that $\mu(C)_j = |b_j^\mu \cap C|$, the number of the i elements in C which lie in the j th block of μ . Since elements of \mathfrak{S}_μ preserve the blocks of μ (by definition), the composition $\mu(C)$ depends only on the orbit $O(\mu, C)$, that is, $\mu(\rho C) = \mu(C)$ for any $\rho \in \mathfrak{S}_\mu$. Let $\mathfrak{S}_{\mu(C)} \subseteq \mathfrak{S}_i$ be the corresponding Young subgroup.

Lemma 4.1. *Given $C \in C(i, r)$ and $\rho \in \mathfrak{S}_\mu$, let $\rho C \in O(\mu, C)$ be the image of C under ρ . Then there exists a unique $\rho_C \in \mathfrak{S}_{\mu(C)}$ such that $\rho \cdot \phi_C = \phi_{\rho_C} \cdot \rho_C$. Conversely, given any $\rho_C \in \mathfrak{S}_{\mu(C)}$ and any $\rho C \in O(\mu, C)$, there exists a $\rho' \in \mathfrak{S}_\mu$ such that $\rho' \cdot \phi_C = \phi_{\rho_C} \cdot \rho_C$ and $\rho' C = \rho C \in O(\mu, C)$.*

Proof. $\rho \cdot \phi_C$ and ϕ_{ρ_C} both map \bar{i} one to one onto ρC . So define $\rho_C \in \mathfrak{S}_i$ by letting $\rho_C(k)$ be the unique element in $\phi_{\rho_C}^{-1}[\rho \cdot \phi_C(k)]$. Then ρ_C is the unique element in \mathfrak{S}_i such that $\rho \cdot \phi_C = \phi_{\rho_C} \cdot \rho_C$, and we need only prove that $\rho_C \in \mathfrak{S}_{\mu(C)}$. So suppose j is in the k th $\mathfrak{S}_{\mu(C)}$ block $b_k^{\mu(C)}$. We must show that $\rho_C(j) \in b_k^{\mu(C)}$. But $j \in b_k^{\mu(C)} \Rightarrow \phi_C(j) \in b_k^\mu \Rightarrow \rho \cdot \phi_C(j) \in b_k^\mu$ (since $\rho \in \mathfrak{S}_\mu$). Then $\phi_{\rho_C} \cdot \rho_C(j) \in b_k^\mu$ which implies $\rho_C(j) \in b_k^{\mu(\rho C)}$. But $\mu(\rho C) = \mu(C)$, so $\rho_C(j) \in b_k^{\mu(C)}$ as desired.

Now take any $\rho_C \in \mathfrak{S}_{\mu(C)}$ and any $\rho C \in O(\mu, C)$. ϕ_C and $\phi_{\rho_C} \cdot \rho_C$ both take the $\mu(C)_j$ elements in the j th block of $\mu(C)$ one to one onto $\mu(C)_j$ elements in the j th block of μ . So there exists $\rho' \in \mathfrak{S}_\mu$ such that $\rho' \cdot \phi_C = \phi_{\rho_C} \cdot \rho_C$. Evidently $\rho' C = \text{image}(\rho' \cdot \phi_C) = \text{image}(\phi_{\rho_C} \cdot \rho_C) = \rho C$. \square

Next, let P_r be the set of all 2^r subsets of \bar{r} . For a composition $\nu \in \Lambda(r, n)$, \mathfrak{S}_ν acts on the right on P_r : if $S \in P_r$ and $\pi \in \mathfrak{S}_\nu$, then $S\pi = \pi^{-1}(S) = \{\pi^{-1}[s] : s \in S\} \in P_r$. Choose a total order on the orbits of \mathfrak{S}_ν acting on P_r and then label the orbits O_i so that $O_1 < O_2 < \dots < O_{N_\nu}$ where N_ν is the number of orbits. Then choose a total order of all the subsets in P_r which is compatible with the ordering of the \mathfrak{S}_ν orbits: $S_{a_i} \in O_{a_i}$ and $O_{a_1} < O_{a_2} \Rightarrow S_{a_1} < S_{a_2}$. We label the subsets S_i so that $S_1 < S_2 < \dots < S_{2^r}$.

Now consider an element $D \in D(M, i, r)$. D consists of i sets in P_r , so $D = \{S_{a_1} < S_{a_2} < \dots < S_{a_i}\}$. Define a composition $\nu(D) \in \Lambda(i, N_\nu)$ by $\nu(D)_j = |D \cap O_j|$, $j = 1, 2, \dots, N_\nu$. Then S_{a_k}, S_{a_l} are in the same orbit O_j if and only if k, l are in the same block of the composition $\nu(D)$, $k, l \in b_j^{\nu(D)}$. \mathfrak{S}_ν acts on the right on $D(M, i, r)$: For $\pi \in \mathfrak{S}_\nu$, $D\pi = \{S_{a_j}\pi : j \in \bar{i}\} = \{\pi^{-1}[S_{a_j}] : j \in \bar{i}\} \in D(M, i, r)$. Since by definition \mathfrak{S}_ν preserves orbits, $|D\pi \cap O_j| = |D \cap O_j|$ for any $\pi \in \mathfrak{S}_\nu$. So the composition $\nu(D\pi) = \nu(D)$ depends only on the orbit $O(\nu, D)$ of D under the action of \mathfrak{S}_ν . Let $\mathfrak{S}_{\nu(D)} \subseteq \mathfrak{S}_i$ be the corresponding Young subgroup.

Lemma 4.2. *Given $D \in D(M, i, r)$ and $\pi \in \mathfrak{S}_\nu$, let $D\pi \in O(\nu, D)$ be the image of D under π . Then there exists a unique $\pi_D \in \mathfrak{S}_{\nu(D)}$ such that $\psi_D \cdot \pi = \pi_D \cdot \psi_{D\pi}$. Conversely, given any $\pi_D \in \mathfrak{S}_{\nu(D)}$ and any $D\pi \in O(\nu, D)$, there exists a $\pi' \in \mathfrak{S}_\nu$ such that $\psi_D \cdot \pi' = \pi_D \cdot \psi_{D\pi}$ and $D\pi' = D\pi \in O(\nu, D)$.*

Proof. Let $D = \{S_{a_1} < S_{a_2} < \dots < S_{a_i}\}$, so $D\pi = \{\pi^{-1}(S_{a_j}) : j \in \bar{i}\}$. Arrange the sets in $D\pi$ in order and define $k(j) \in \bar{i}$ such that $\pi^{-1}(S_{a_j})$ is the $k(j)$ th set in the sequence. Then $\psi_D \cdot \pi$ maps elements in $\pi^{-1}(S_{a_j})$ to j , while $\psi_{D\pi}$ maps elements in $\pi^{-1}(S_{a_j})$ to $k(j)$. Define $\pi_D \in \mathfrak{S}_i$ by $\pi_D(k(j)) = j$, $j \in \bar{i}$. Then π_D is the unique element in \mathfrak{S}_i such that $\psi_D \cdot \pi = \pi_D \cdot \psi_{D\pi}$, and it remains to show that $\pi_D \in \mathfrak{S}_{\nu(D)}$. For this, we must show that j and $k(j)$ are always in the same block of the composition $\nu(D)$. But $D\pi$ and D contain the same

number of subsets in each orbit of \mathfrak{S}_ν . So if S_{a_j} and $\pi^{-1}(S_{a_j})$ are both in the l th orbit of \mathfrak{S}_ν , then their indices j and $k(j)$ are both in the l th block of $\nu(D)$.

Now take any $\pi_D \in \mathfrak{S}_{\nu(D)}$ and any $D\pi \in O(\nu, D)$. Recall that $\bar{r} \supseteq \bigcup_{k \in \bar{i}} \pi^{-1}(S_{a_k})$ and that the sets $\pi^{-1}(S_{a_k})$ are pairwise disjoint, so we can define a $\pi' \in \mathfrak{S}_r$ by defining $\pi'| \pi^{-1}(S_{a_k})$ for each k . Suppose k is in the j th block of $\nu(D)$. Then both $(\psi_D)^{-1}(k) = S_{a_k}$ and $(\pi_D \cdot \psi_{D\pi})^{-1}(k) = \pi^{-1}(S_{a_{m(k)}})$ (for some $m(k)$) are in the j th orbit of \mathfrak{S}_ν . Define $\pi' \in \mathfrak{S}_r$ by $\pi'|(\pi^{-1}(S_{a_{m(k)}})) = \sigma_k|(\pi^{-1}(S_{a_{m(k)}}))$ where $\sigma_k \in \mathfrak{S}_\nu$ maps $\pi^{-1}(S_{a_{m(k)}})$ one to one onto S_{a_k} . Then $(\psi_D \cdot \pi')^{-1}(k) = (\pi')^{-1}(\psi_D^{-1}(k)) = (\pi')^{-1}(S_{a_k}) = \pi^{-1}(S_{a_{m(k)}}) = (\pi_D \cdot \psi_{D\pi})^{-1}(k)$ for all k . So $\psi_D \cdot \pi' = \pi_D \cdot \psi_{D\pi}$ and it remains to show that $\pi' \in \mathfrak{S}_\nu$. It suffices to show that if l is in the j th block of ν then $\pi'(l)$ is also in the j th block. But any l is in a unique $\pi^{-1}(S_{a_k})$, so $\pi'(l) = \sigma_k(l)$ is in fact in the j th block since $\sigma_k \in \mathfrak{S}_\nu$. \square

Consider compositions $\mu, \nu \in \Lambda(r, n)$ and an element $\alpha \in M$ of index i . There exist unique $\sigma_\alpha \in \mathfrak{S}_i$, $C_\alpha \in C(i, r)$, $D_\alpha \in D(M, i, r)$ such that $\alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha}$. For $C = \rho C_\alpha \in O(\mu, C_\alpha)$, define

$$\phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha} = \{ \phi_C \cdot \kappa \cdot \sigma_\alpha \circ \psi_{D_\alpha} : \kappa \in \mathfrak{S}_{\mu(C)} \} \subseteq M.$$

Similarly, for $D = D_\alpha \pi \in O(\nu, D_\alpha)$, define

$$\phi_{C_\alpha} \circ \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_D = \{ \phi_{C_\alpha} \circ \sigma_\alpha \cdot \gamma \cdot \psi_D : \gamma \in \mathfrak{S}_{\nu(D)} \}.$$

Finally, for $C \in O(\mu, C_\alpha)$, $D \in O(\nu, D_\alpha)$, define

$$\phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_D = \{ \phi_C \cdot \kappa \cdot \sigma_\alpha \cdot \gamma \cdot \psi_D : \kappa \in \mathfrak{S}_{\mu(C)}, \gamma \in \mathfrak{S}_{\nu(D)} \}.$$

Proposition 4.1. *For compositions $\mu, \nu \in \Lambda(r, n)$ and an element $\alpha = \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \in M$ of index i ,*

(a) *For $C_1, C_2, C \in O(\mu, C_\alpha)$, $D_1, D_2, D \in O(\nu, D_\alpha)$,*

$$(\phi_{C_1} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_2}) = \emptyset$$

unless $C_1 = C_2$ and $D_1 = D_2$. The double coset $\mathfrak{S}_\mu \alpha \mathfrak{S}_\nu$ is a disjoint union

$$\mathfrak{S}_\mu \alpha \mathfrak{S}_\nu = \bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_D.$$

(b) *For $C_1, C_2 \in O(\mu, C_\alpha)$,*

$$(\phi_{C_1} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha}) \cap (\phi_{C_2} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha}) = \emptyset$$

unless $C_1 = C_2$.

$$\mathfrak{S}_\mu \alpha = \bigcup_{C \in O(\mu, C_\alpha)} \phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \circ \psi_{D_\alpha},$$

a disjoint union.

(c) For $D_1, D_2 \in O(\nu, D_\alpha)$,

$$(\phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_2}) = \emptyset$$

unless $D_1 = D_2$.

$$\alpha \mathfrak{S}_\nu = \bigcup_{D \in O(\nu, D_\alpha)} \phi_{C_\alpha} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \circ \psi_D,$$

a disjoint union.

Proof. For part (a), if

$$\beta \in (\phi_{C_1} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1})$$

then $C_1 = C_\beta = C_2$ and $D_1 = D_\beta = D_2$, so

$$(\phi_{C_1} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1}) \cap (\phi_{C_2} \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_{D_1}) = \emptyset$$

unless $C_1 = C_2$ and $D_1 = D_2$.

If $\beta \in \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu$, then $\beta = \rho \circ \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \circ \pi$ for some $\rho \in \mathfrak{S}_\mu$, $\pi \in \mathfrak{S}_\nu$. By lemmas 4.1 and 4.2, there exist $\rho_C \in \mathfrak{S}_{\mu(C)}$, $\pi_D \in \mathfrak{S}_{\nu(D)}$ such that

$$\beta = \phi_{\rho C_\alpha} \circ \rho_C \circ \sigma_\alpha \circ \pi_D \circ \psi_{D_\alpha} \pi \in \phi_{\rho C_\alpha} \cdot \mathfrak{S}_{\mu(C)} \circ \sigma_\alpha \circ \mathfrak{S}_{\nu(D)} \cdot \psi_{D_\alpha} \pi.$$

So $\mathfrak{S}_\mu \alpha \mathfrak{S}_\nu \subseteq \bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_D$. On the other hand, if

$$\beta = \phi_{\rho C_\alpha} \circ \rho_C \circ \sigma_\alpha \circ \pi_D \circ \psi_{D_\alpha} \pi \in \phi_{\rho C_\alpha} \cdot \mathfrak{S}_{\mu(C)} \circ \sigma_\alpha \circ \mathfrak{S}_{\nu(D)} \cdot \psi_{D_\alpha} \pi,$$

then by lemmas 4.1 and 4.2 there exist $\rho' \in \mathfrak{S}_\mu$, $\pi' \in \mathfrak{S}_\nu$ such that

$$\beta = \rho' \cdot \phi_{C_\alpha} \circ \sigma_\alpha \circ \psi_{D_\alpha} \cdot \pi' = \rho' \alpha \pi' \in \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu.$$

So $\bigcup_{C \in O(\mu, C_\alpha), D \in O(\nu, D_\alpha)} \phi_C \cdot \mathfrak{S}_{\mu(C)} \cdot \sigma_\alpha \cdot \mathfrak{S}_{\nu(D)} \cdot \psi_D \subseteq \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu$, completing the proof of part (a).

(b) follows from (a) by taking $D_1 = D_2 = D_\alpha$ and ν to be the composition $\nu_i = 1, \forall i$, so $\mathfrak{S}_\nu = \mathfrak{S}_{\nu(D)} = \{1\}$. Similarly, (c) follows from (a) by taking $C_1 = C_2 = C_\alpha$ and μ to be the composition $\mu_i = 1, \forall i$, so $\mathfrak{S}_\mu = \mathfrak{S}_{\mu(C)} = \{1\}$. \square

Any double coset $\mathbf{D} = \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu \in {}_\mu M_\nu$ has a well defined index i since any $\beta \in \mathbf{D}$ has the same index as α . We also have $O(\mu, C_\beta) = O(\mu, C_\alpha)$ for any $\beta \in \mathbf{D}$, so \mathbf{D} has a well defined orbit $O(\mu, \mathbf{D}) = O(\mu, C_\alpha)$ for the action of \mathfrak{S}_μ on $C(i, r)$. There is also a well defined composition $\mu(\mathbf{D})$ and a Young subgroup $\mathfrak{S}_{\mu(\mathbf{D})}$ where $\mu(\mathbf{D}) = \mu(C)$ for any $C \in O(\mu, \mathbf{D})$. Both $\mu(\mathbf{D})$ and $\mathfrak{S}_{\mu(\mathbf{D})}$ depend only on the orbit $O(\mu, \mathbf{D})$. For a double coset $\mathbf{D} = \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu \in {}_\mu M_\nu$, a left coset $\mathbf{C} \subseteq \mathfrak{S}_\mu \alpha \mathfrak{S}_\nu$ has the form $\mathbf{C} = \mathfrak{S}_\mu \alpha \cdot \pi = \bigcup_{C \in O(\mu, C_\alpha)} \phi_C \cdot \mathfrak{S}_{\mu(C_\alpha)} \cdot \sigma_\alpha \circ \psi_{D_\alpha} \cdot \pi$

for some $\pi \in \mathfrak{S}_\nu$, where the union is disjoint by proposition 4.1. Then $n_L(\mathbf{D}) = |\mathbf{C}| = |O(\mu, C_\alpha)| \cdot |\mathfrak{S}_{\mu(C_\alpha)}| = |O(\mu, \mathbf{D})| \cdot |\mathfrak{S}_{\mu(\mathbf{D})}|$, which depends only on the orbit $O_\mu = O(\mu, \mathbf{D}) = O(\mu, C_\alpha)$. So we can define $n_L(O_\mu) = n_L(\mathbf{D})$ for any \mathbf{D} with $O_\mu = O(\mu, \mathbf{D})$.

Similarly, \mathbf{D} has a well defined orbit $O(\nu, \mathbf{D}) = O(\nu, D_\alpha)$ for the action of \mathfrak{S}_ν on $D(M, i, r)$ and there are a composition $\nu(\mathbf{D})$ and a Young subgroup $\mathfrak{S}_{\nu(\mathbf{D})}$ where $\nu(\mathbf{D}) = \nu(D)$ for any $D \in O(\nu, \mathbf{D})$. Then $\nu(\mathbf{D})$, $\mathfrak{S}_{\nu(\mathbf{D})}$, and $n_R(\mathbf{D}) = |O(\nu, \mathbf{D})| \cdot |\mathfrak{S}_{\nu(\mathbf{D})}|$ depend only on the orbit $O(\nu, \mathbf{D})$ and we can define $n_R(O_\nu) = n_R(\mathbf{D})$ for any \mathbf{D} with $O_\nu = O(\nu, \mathbf{D})$.

Given an orbit pair $(O_\mu, O_\nu) \in \mathbf{O}(\mu, \nu, M, i)$, define compositions $\mu(O_\mu)$ and $\nu(O_\nu)$ of i and corresponding Young subgroups $\mathfrak{S}_{\mu(O_\mu)}$, $\mathfrak{S}_{\nu(O_\nu)}$, where $\mu(O_\mu) = \mu(C)$ for any $C \in O_\mu$ and $\nu(O_\nu) = \nu(D)$ for any $D \in O_\nu$. If ${}^{\mu(O_\mu)}B^{\nu(O_\nu)} \subseteq \mathbb{Z}[\mathfrak{S}_i]$ is the \mathbb{Z} -submodule of $B = \mathbb{Z}[\mathfrak{S}_i]$ which is invariant under the action of $\mathfrak{S}_{\mu(O_\mu)}$ on the left and $\mathfrak{S}_{\nu(O_\nu)}$ on the right, then ${}^{\mu(O_\mu)}B^{\nu(O_\nu)}$ is a free \mathbb{Z} -module with basis $\{X(\mathfrak{S}_{\mu(O_\mu)}\sigma\mathfrak{S}_{\nu(O_\nu)}) : \sigma \in \mathfrak{S}_i\}$. Define a \mathbb{Z} -linear map $\Phi(O_\mu, O_\nu) : {}^{\mu(O_\mu)}B^{\nu(O_\nu)} \rightarrow {}^{O_\mu}A^{O_\nu}$ by

$$\Phi(O_\mu, O_\nu)(x) = \sum_{C \in O_\mu} \sum_{D \in O_\nu} \phi_C \circ x \circ \psi_D.$$

By proposition 4.1, $\Phi(O_\mu, O_\nu)$ is an isomorphism of free \mathbb{Z} -modules taking the basis elements $X(\mathfrak{S}_{\mu(O_\mu)}\sigma\mathfrak{S}_{\nu(O_\nu)})$ for ${}^{\mu(O_\mu)}B^{\nu(O_\nu)}$ one to one onto the basis elements $X(\mathfrak{S}_\mu\alpha\mathfrak{S}_\nu)$ for ${}^{O_\mu}A^{O_\nu}$ where $\alpha = \phi_C \circ \sigma \circ \psi_D$ for any $C \in O_\mu, D \in O_\nu$. Now as a \mathbb{Z} -module, ${}^{\mu(O_\mu)}B^{\nu(O_\nu)}$ can be identified with a direct summand of the standard Schur algebra $S_{\mathbb{Z}}(r, n)$. The standard cellular basis for the Schur algebra $S_{\mathbb{Z}}(r, n)$ then yields a basis $\{{}_S C_T^\lambda\}$ for ${}^{\mu(O_\mu)}B^{\nu(O_\nu)}$ where λ is a partition of i , S is a semistandard λ tableau of type $\mu(O_\mu)$ and T is a semistandard λ tableau of type $\nu(O_\nu)$. Then $\{\Phi(O_\mu, O_\nu)({}_S C_T^\lambda)\}$ gives a basis $B(O_\mu, O_\nu)$ for ${}^{O_\mu}A^{O_\nu}$. These piece together to give a basis for A which turns out to be a cell basis for $A_L^{\mathbb{Z}}$ or $A_R^{\mathbb{Z}}$.

Let $\Lambda = \bigcup_{i \in I(M)} \Lambda(i)$ with the same partial order as for the cell algebra $\mathbb{Z}[M]$ of section 7. For $\lambda \in \Lambda(i) \subset \Lambda$ define

$$L(\lambda) = \{O_\mu, S : \mu \in \Lambda(r, n), O_\mu \in \mathbf{O}(\mu, C(i, r)), S \in SSt(\lambda, \mu(O_\mu))\}$$

where $SSt(\lambda, \mu(O_\mu))$ is the set of semistandard λ tableaux of type $\mu(O_\mu)$ and define

$$R(\lambda) = \{O_\nu, T : \nu \in \Lambda(r, n), O_\nu \in \mathbf{O}(\nu, D(M, i, r)), T \in SSt(\lambda, \nu(O_\nu))\}$$

where $SSt(\lambda, \nu(O_\nu))$ is the set of semistandard λ tableaux of type $\nu(O_\nu)$. Then for $\lambda \in \Lambda$, $(O_\mu, S) \in L(\lambda)$, $(O_\nu, T) \in R(\lambda)$, define ${}_{(O_\mu, S)}C_{(O_\nu, T)}^\lambda = \Phi(O_\mu, O_\nu)({}_S C_T^\lambda)$. As just mentioned, elements of this type provide a \mathbb{Z} -basis for each direct summand ${}^{O_\mu}A^{O_\nu}$ and hence for all of \bar{A} .

Note that if M contains the zero map z where $z(j) = 0$ for all j , then Λ contains the empty partition λ^0 of index 0. For any partitions μ, ν the double

coset $\mathfrak{S}_\mu z \mathfrak{S}_\nu = \{z\} \subseteq {}^\mu A^\nu$. $C(0, r) = D(M, 0, r) = \emptyset$, so for each partition there is one orbit O_μ or O_ν . Then $L(\lambda^0)$ contains one element (O_μ, \emptyset) for each partition μ , and similarly for $R(\lambda^0)$. Then for each pair of partitions μ, ν our basis contains an element ${}_{(O_\mu, \emptyset)} C_{(O_\nu, \emptyset)}^{\lambda^0} = z \in {}^\mu A^\nu$.

We now show that the basis just described is a cell basis. We first check that these basis elements have the left and right cell algebra properties (i) and (ii) for the “ordinary” product in \bar{A} . We then check that the properties also hold for the products $*_L$ and $*_R$ in $A_L^\mathbb{Z}$ and $A_R^\mathbb{Z}$.

Notice that each basis element ${}_{(O_\mu, S)} C_{(O_\nu, T)}^\lambda$ for \bar{A} is a sum of basis elements in the cell algebra $A = \mathbb{Z}[M]$ of section 7 of the form ${}_{C, s} C_{t, D}^\lambda$ for the same λ (where s, t are standard tableaux of type S, T , $C \in O_\mu$, $D \in O_\nu$). As in section 7, let $A^\lambda, \hat{A}^\lambda$ be the ideals in the cell algebra $A = \mathbb{Z}[M]$ and let $\bar{A}^\lambda, \hat{\bar{A}}^\lambda$ be the corresponding submodules of \bar{A} (spanned by basis elements ${}_{(O_\mu, S)} C_{(O_\nu, T)}^\kappa$ with $\kappa \geq \lambda$ or $\kappa > \lambda$ respectively). Then $\hat{A}^\lambda \cap {}^\mu A^\nu = \hat{\bar{A}}^\lambda \cap {}^\mu A^\nu$ for any λ, μ, ν .

Lemma 4.3. *Let ${}_{(O_{\mu_i}, S_i)} C_{(O_{\nu_i}, T_i)}^{\lambda_i} \in {}^{O_{\mu_i}} A^{O_{\nu_i}}$ for $i = 1, 2$. Then in \bar{A} , ${}_{(O_{\mu_1}, S_1)} C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)} C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\mu', S'} r \cdot {}_{(O_{\mu'}, S')} C_{(O_{\nu_2}, T_2)}^{\lambda_2} \pmod{\hat{A}^{\lambda_2}}$ where the coefficients $r \in \mathbb{Z}$ are independent of O_{ν_2} and T_2 .*

Proof. Write ${}_{(O_{\mu_2}, S_2)} C_{(O_{\nu_2}, T_2)}^{\lambda_2}$ as a sum of terms ${}_{C, s} C_{t, D}^{\lambda_2}$ where t is a standard tableau of type T and $D \in O_{\nu_2}$. Then using property (i) for the cell algebra A , we have ${}_{(O_{\mu_1}, S_1)} C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)} C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{C', s'} r \cdot {}_{C', s'} C_{t, D}^{\lambda_2} \pmod{\hat{A}^{\lambda_2}}$ where the coefficients r are independent of D and t and therefore of O_{ν_2} and T_2 . Also ${}_{(O_{\mu_1}, S_1)} C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)} C_{(O_{\nu_2}, T_2)}^{\lambda_2} \in {}^{\mu_1} A^{\nu_2}$. So the terms ${}_{C', s'} C_{t, D}^{\lambda_2}$ must regroup into a linear combination of terms of the form ${}_{(O_{\mu'}, S')} C_{(O_{\nu_2}, T_2)}^{\lambda_2}$. Then using $\hat{A}^{\lambda_2} \cap {}^{\mu_1} A^{\nu_2} = \hat{\bar{A}}^{\lambda_2} \cap {}^{\mu_1} A^{\nu_2}$ gives the result. \square

Since the ${}_{(O_\mu, S)} C_{(O_\nu, T)}^\lambda$ form a basis for \bar{A} , linearity gives the following corollary.

Corollary 4.1. *For any $x \in \bar{A}$,*

$$x \cdot {}_{(O_\mu, S)} C_{(O_\nu, T)}^\lambda = \sum_{\mu', S'} r \cdot {}_{(O_{\mu'}, S')} C_{(O_\nu, T)}^\lambda \pmod{\hat{A}^\lambda}$$

where $r = r(x, \mu, S, \mu', S')$ is independent of O_ν, T .

Similar arguments give the following results.

Lemma 4.4. *Let ${}_{(O_{\mu_i}, S_i)} C_{(O_{\nu_i}, T_i)}^{\lambda_i} \in {}^{O_{\mu_i}} A^{O_{\nu_i}}$ for $i = 1, 2$. Then in \bar{A} , ${}_{(O_{\mu_1}, S_1)} C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)} C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\nu', T'} r \cdot {}_{(O_{\mu_1}, S_1)} C_{(O_{\nu'}, T')}^{\lambda_1} \pmod{\hat{A}^{\lambda_1}}$ where the coefficients $r \in \mathbb{Z}$ are independent of O_{μ_1} and S_1 .*

Corollary 4.2. For any $x \in \bar{A}$,

$$({}_{O_\mu, S})C_{(O_\nu, T)}^\lambda \cdot x = \sum_{\nu', T'} r \cdot ({}_{O_\mu, S})C_{(O_{\nu'}, T')}^\lambda \mod \hat{A}^\lambda$$

where $r = r(x, \nu, T, \nu', T')$ is independent of O_μ, S .

We now transfer our results to $A_L^\mathbb{Z}$ and $A_R^\mathbb{Z}$. We need the following lemma.

Lemma 4.5. Let $B = \cup B(O_\mu, O_\nu)$ be a basis for \bar{A} where each $B(O_\mu, O_\nu)$ is a basis for the direct summand ${}^{O_\mu}A^{O_\nu}$. For $b \in B(O_\mu, O_\nu)$ define $n_L(b) = n_L(O_\mu)$ and $n_R(b) = n_R(O_\nu)$. Assume that $b_1 b_2 = \sum_{b \in B} c(b_1, b_2, b) b$ with structure constants $c(b_1, b_2, b) \in \mathbb{Z}$ (using the “ordinary” product in \bar{A}). Then $b_1 *_L b_2 = \sum_{b \in B} \frac{n_L(b)}{n_L(b_1)n_L(b_2)} c(b_1, b_2, b) b$ and $b_1 *_R b_2 = \sum_{b \in B} \frac{n_R(b)}{n_R(b_1)n_R(b_2)} c(b_1, b_2, b) b$.

Proof. The result is true by definition for the standard basis $\{b_D = X(D)\}$. Each $b \in B(O_\mu, O_\nu)$ is a linear combination of standard basis vectors $b_D \in B(O_\mu, O_\nu)$ and each $b_D \in {}^{O_\mu}A^{O_\nu}$ is a linear combination of the new $b \in B(O_\mu, O_\nu)$. Then since the values $n_L(b)$, $n_R(b)$ depend only on the orbits O_μ, O_ν , the result for the new basis follows by linearity. \square

Lemma 4.6. Let $({}_{O_{\mu_i}, S_i})C_{(O_{\nu_i}, T_i)}^{\lambda_i} \in {}^{O_{\mu_i}}A^{O_{\nu_i}}$ for $i = 1, 2$. Assume $\nu_1 = \mu_2$.

(a) In $A_L^\mathbb{Z}$,

$$({}_{O_{\mu_1}, S_1})C_{(O_{\nu_1}, T_1)}^{\lambda_1} *_L ({}_{O_{\mu_2}, S_2})C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\mu', S'} r \cdot ({}_{O_{\mu'}, S'})C_{(O_{\nu_2}, T_2)}^{\lambda_2} \mod \hat{A}^{\lambda_2}$$

where the coefficients $r \in \mathbb{Z}$ are independent of O_{ν_2} and T_2 .

(b) In $A_L^\mathbb{Z}$,

$$({}_{O_{\mu_1}, S_1})C_{(O_{\nu_1}, T_1)}^{\lambda_1} *_L ({}_{O_{\mu_2}, S_2})C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\nu', T'} r \cdot ({}_{O_{\mu_1}, S_1})C_{(O_{\nu'}, T')}^{\lambda_1} \mod \hat{A}^{\lambda_1}$$

where the coefficients $r \in \mathbb{Z}$ are independent of O_{μ_1} and S_1 .

(c) In $A_R^\mathbb{Z}$,

$$({}_{O_{\mu_1}, S_1})C_{(O_{\nu_1}, T_1)}^{\lambda_1} *_R ({}_{O_{\mu_2}, S_2})C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\mu', S'} r \cdot ({}_{O_{\mu'}, S'})C_{(O_{\nu_2}, T_2)}^{\lambda_2} \mod \hat{A}^{\lambda_2}$$

where the coefficients $r \in \mathbb{Z}$ are independent of O_{ν_2} and T_2 .

(d) In $A_R^\mathbb{Z}$,

$$({}_{O_{\mu_1}, S_1})C_{(O_{\nu_1}, T_1)}^{\lambda_1} *_R ({}_{O_{\mu_2}, S_2})C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\nu', T'} r \cdot ({}_{O_{\mu_1}, S_1})C_{(O_{\nu'}, T')}^{\lambda_1} \mod \hat{A}^{\lambda_1}$$

where the coefficients $r \in \mathbb{Z}$ are independent of O_{μ_1} and S_1 .

Proof. Notice that a basis element $b = {}_{(O_\mu, S)}C_{(O_\nu, T)}^\lambda$ is in ${}^{O_\mu}A^{O_\nu}$, so in the notation of lemma 4.5 we have $n_L(b) = n_L(O_\mu)$, $n_R(b) = n_R(O_\nu)$.

For part (a), lemma 4.3 gives

$${}_{(O_{\mu_1}, S_1)}C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)}C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\mu', S'} r' \cdot {}_{(O_{\mu'}, S')}C_{(O_{\nu_2}, T_2)}^{\lambda_2} \mod \hat{A}^{\lambda_2}$$

where the coefficients $r' \in \mathbb{Z}$ are independent of O_{ν_2} and T_2 . Then lemma 4.5 gives

$$\begin{aligned} & {}_{(O_{\mu_1}, S_1)}C_{(O_{\nu_1}, T_1)}^{\lambda_1} *L {}_{(O_{\mu_2}, S_2)}C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \\ & \sum_{\mu', S'} \frac{n_L(O_{\mu'})}{n_L(O_{\mu_1})n_L(O_{\mu_2})} \cdot r' \cdot {}_{(O_{\mu'}, S')}C_{(O_{\nu_2}, T_2)}^{\lambda_2} \mod \hat{A}^{\lambda_2}. \end{aligned}$$

Observing that $r = \frac{n_L(O_{\mu'})}{n_L(O_{\mu_1})n_L(O_{\mu_2})} \cdot r'$ is independent of O_{ν_2} and T_2 gives the result (a).

For (b), lemma 4.4 gives

$${}_{(O_{\mu_1}, S_1)}C_{(O_{\nu_1}, T_1)}^{\lambda_1} \cdot {}_{(O_{\mu_2}, S_2)}C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \sum_{\nu', T'} r' \cdot {}_{(O_{\mu_1}, S_1)}C_{(O_{\nu'}, T')}^{\lambda_1} \mod \hat{A}^{\lambda_1}$$

where the coefficients $r' \in \mathbb{Z}$ are independent of O_{μ_1} and S_1 . Then lemma 4.5 gives

$$\begin{aligned} & {}_{(O_{\mu_1}, S_1)}C_{(O_{\nu_1}, T_1)}^{\lambda_1} *L {}_{(O_{\mu_2}, S_2)}C_{(O_{\nu_2}, T_2)}^{\lambda_2} = \\ & \sum_{\nu', T'} \frac{n_L(O_{\mu_1})}{n_L(O_{\mu_1})n_L(O_{\mu_2})} \cdot r' \cdot {}_{(O_{\mu_1}, S_1)}C_{(O_{\nu'}, T')}^{\lambda_1} \mod \hat{A}^{\lambda_1}. \end{aligned}$$

Then $r = \frac{n_L(O_{\mu_1})}{n_L(O_{\mu_1})n_L(O_{\mu_2})} \cdot r' = \frac{1}{n_L(O_{\mu_2})} \cdot r'$ is independent of O_{μ_1} and S_1 and the result (b) follows.

Parts (c) and (d) are proved similarly. \square

Proposition 4.2. $\left\{ {}_{(O_\mu, S)}C_{(O_\nu, T)}^\lambda \right\}$ is a cell basis for both $A_L^\mathbb{Z}$ and $A_R^\mathbb{Z}$, which are therefore cell algebras.

Proof. We have shown the $\left\{ {}_{(O_\mu, S)}C_{(O_\nu, T)}^\lambda \right\}$ form a basis and the multiplication rules (i) and (ii) for a cell algebra then follow at once by linearity from lemma 4.6. \square

Corollary 4.3. For any domain R , the left and right generalized Schur algebras $LGS_R(M, \mathbf{G}) = R \otimes_{\mathbb{Z}} A_L^\mathbb{Z}$ and $RGS_R(M, \mathbf{G}) = R \otimes_{\mathbb{Z}} A_R^\mathbb{Z}$ are cell algebras with a cell basis $\left\{ {}_{(O_\mu, S)}C_{(O_\nu, T)}^\lambda \right\}$.

5 Irreducible modules for generalized Schur algebras

The cell basis $\left\{_{(O_\mu, S)} C_{(O_\nu, T)}^\lambda\right\}$ for the cell algebra $A_L^\mathbb{Z}$ or $A_R^\mathbb{Z}$ found above depends on the choice of an ordering of the orbits of \mathfrak{S}_ν acting on P_r and of an ordering of the subsets in P_r compatible with the ordering of the orbits. We now choose orderings which will simplify the calculations of the brackets in these cell algebras.

For $d \in P_r$, $|d| = i$, define an increasing string of i integers, $s(d)$, to be the i elements of d arranged in ascending order. Then define a non-decreasing string of i of positive integers, $s(\nu, d)$, by replacing each $x \in s(d)$ by $b(x)$, where x is in the $b(x)^{th}$ block $b_\nu^{b(x)}$ of the composition \mathfrak{S}_ν . Finally, define a string of r non-negative integers, $\bar{s}(\nu, d)$, by adding $r - i$ zeroes to the end of $s(\nu, d)$. Note that $\bar{s}(\nu, d)$ depends only on the \mathfrak{S}_ν -orbit of d . In fact, $\bar{s}(\nu, d) = \bar{s}(\nu, d') \Leftrightarrow d, d'$ are in the same \mathfrak{S}_ν -orbit. We then get a total ordering of the \mathfrak{S}_ν -orbits by ordering the corresponding strings $\bar{s}(\nu, d)$ lexicographically: if $\bar{s}(\nu, d)_j$ represents the j th element of the string, then $\bar{s}(\nu, d) < \bar{s}(\nu, d') \Leftrightarrow$ for some J between 1 and r we have $\bar{s}(\nu, d)_j = \bar{s}(\nu, d')_j$ for all $j < J$, while $\bar{s}(\nu, d)_J < \bar{s}(\nu, d')_J$.

We then define our order on P_r by : $d < d'$ if (1) $\bar{s}(\nu, d) < \bar{s}(\nu, d')$ or (2) $\bar{s}(\nu, d) = \bar{s}(\nu, d')$ (so d, d' are in the same \mathfrak{S}_ν -orbit) and $s(d) < s(d')$ in lexicographical order.

Note the following special cases of our ordering:

The smallest set in P_r is the empty set \emptyset .

If $\{a\}, \{b\}$ are one element sets in P_r , then $\{a\} < \{b\} \Leftrightarrow a < b$.

If $\{a\}, d \in P_r$ and d has more than one element with smallest element b , then $\{a\} < d$ if the ν - block containing $a \leq$ the ν - block containing b , while $\{a\} > d$ if the ν - block containing $a >$ the ν - block containing b .

We will assume our cell bases are chosen with respect to these orderings.

In this section we assume that $R = k$ is a field. We will write $S_L(M, k)$ for the left generalized Schur algebra $LGS_k(M, \mathbf{G}) = k \otimes A_L^\mathbb{Z}$ and $S_R(M, k)$ for the right generalized Schur algebra $RGS_k(M, \mathbf{G}) = k \otimes A_R^\mathbb{Z}$. For either of these algebras, if $\lambda \in \Lambda$ then $\lambda \in \Lambda(i, n)$ for some i with $i \leq r \leq n$. Recall that in these cell algebras Λ_0 is the subset of Λ consisting of λ for which the bracket $\langle_{O_\mu, S} C^\lambda, C_{O_\nu, T}^\lambda \rangle$ is not identically zero. By corollary 2.1, there is one isomorphism class of irreducible modules for each $\lambda \in \Lambda_0$. We will determine Λ_0 when $M = \mathcal{T}_r$ or when M contains the rook monoid \mathfrak{R}_r .

Theorem 5.1. *Let k be a field of characteristic 0 and let $M = \mathcal{T}_r$. Then $\Lambda_0 = \Lambda$ for both $S_L(\tau_r, k)$ and $S_R(\tau_r, k)$. Both $S_L(\tau_r, k)$ and $S_R(\tau_r, k)$ are quasi-hereditary algebras.*

Proof. Take any $\lambda \in \Lambda$ with $\text{index}(\lambda) = i > 0$. Let k be the largest index such that $\lambda_k > 0$, so $\lambda_j = 0$, $j > k$. Let μ be the partition of r where

$$\mu_j = \begin{cases} \lambda_j & \text{if } j \leq k \\ 1 & \text{if } j = k+1, k+2, \dots, k+(r-i) \end{cases}.$$

Let $C = \{1, 2, \dots, i\}$. Then $\mu(C) = \lambda \in \Lambda(i, n)$. Let S be the semistandard λ -tableau of type $\mu(C)$ where $S_{j,l} = j$, $1 \leq l \leq \lambda_j$, $j = 1, 2, \dots, k$. There is only one standard λ -tableau of type S , namely $s = id$, the identity in \mathfrak{S}_i . The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \bar{i} \rightarrow \bar{r}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \dots, i$.

Next let $D = \{\{1\}, \{2\}, \dots, \{i-1\}, \{i, i+1, i+2, \dots, r\}\}$. Then $\mu(D) \in \Lambda(i, n)$ is given by

$$\mu(D)_j = \begin{cases} \lambda_j & \text{if } j < k \\ \lambda_k - 1 & \text{if } j = k \\ 1 & \text{if } j = k + 1 \\ 0 & \text{if } j > k + 1 \end{cases}.$$

Let T be the semistandard λ -tableau of type $\mu(D)$ where

$$T_{j,l} = \begin{cases} j & \text{for } 1 \leq l \leq \lambda_j, j = 1, 2, \dots, k-1 \\ k & \text{for } 1 \leq l \leq \lambda_k - 1, j = k \\ k+1 & \text{for } l = \lambda_k, j = k \end{cases}.$$

There is only one standard λ -tableau of type T , namely $t = id$, the identity in \mathfrak{S}_i . Let b_μ^k be the k th block in the partition μ . For $a \in b_\mu^k$, define $D_a = \{\{1\}, \{2\}, \dots, \{i-1\}, \{i\}, \{a, i+1, i+2, \dots, r\}\} - \{\{a\}\}$. Then the orbit $O(D, \mu) = \{D_a : a \in b_\mu^k\}$ and $\#O(D, \mu) = |b_\mu^k| = \mu_k = \lambda_k$. Also $\psi_{D_a} : \bar{r} \rightarrow \bar{i}$ is given by

$$\psi_{D_a}(j) = \begin{cases} j & \text{if } j < a \\ j-1 & \text{if } a < j \leq i \\ i & \text{if } j = a \text{ or } j > i \end{cases}.$$

Then $\psi_{D_a} \circ \phi_C : \bar{i} \rightarrow \bar{i}$ is a cyclic permutation $\sigma_a = (a, i, i-1, i-2, \dots, a+1) \in \mathfrak{S}_\lambda$.

We have ${}_S C_T^\lambda = {}_S C_t^\lambda = id \cdot r_\lambda \cdot id = r_\lambda$, so

$${}_S C_T^\lambda \circ \psi_{D_a} \circ \phi_C \circ {}_S C_T^\lambda = r_\lambda \cdot \sigma_a \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) r_\lambda.$$

Then writing $b = {}_{O(C, \mu), S} C_{O(D, \mu), T}^\lambda = \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{a \in b_\mu^k} \psi_{D_a}$, compute

$$\begin{aligned} b \cdot b &= \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{a \in b_\mu^k} \psi_{D_a} \cdot \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{a \in b_\mu^k} \psi_{D_a} \\ &= \lambda_k \cdot \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \sum_{a \in b_\mu^k} \psi_{D_a} \\ &= \lambda_k \cdot o(\mathfrak{S}_\lambda) \cdot b. \end{aligned}$$

Then

$$b *_L b = \lambda_k \cdot o(\mathfrak{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b) n_L(b)} \cdot b = \lambda_k \cdot o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_L(b)} \cdot b = \lambda_k \cdot b$$

where

$$n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\mathfrak{S}_{\mu(C)}) = 1 \cdot o(\mathfrak{S}_\lambda).$$

Similarly,

$$b *_R b = \lambda_k \cdot o(\mathfrak{S}_\lambda) \cdot \frac{n_R(b)}{n_R(b) n_R(b)} \cdot b = \lambda_k \cdot o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_R(b)} \cdot b = \lambda_k \cdot b$$

where

$$n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(\mathfrak{S}_{\mu(D)}) = \lambda_k \cdot o(\mathfrak{S}_\lambda) / \lambda_k.$$

Then computing the bracket (in either $S_L(\tau_r, k)$ or $S_R(\tau_r, k)$) we find that $\langle C_{O(D, \mu), T}^\lambda, O(C, \mu), S C^\lambda \rangle = \lambda_k \neq 0$ (since characteristic of k is 0). So $\lambda \in \Lambda_0$.

By corollary 2.1, a cell algebra with $\Lambda_0 = \Lambda$ is quasi-hereditary, so the proof is complete. \square

Now assume k is a field of characteristic $p > 0$. For a partition $\lambda \in \Lambda(i)$, $1 \leq i \leq r$, define an integer $k(p, \lambda) \geq 0$ to be the highest power of p which divides λ_j for every j . (So for all j , $p^{k(p, \lambda)} \mid \lambda_j$, while for at least one j , $p^{k(p, \lambda)+1}$ does not divide λ_j .) Then define $\Lambda_p = \{\lambda \in \Lambda : p^{k(p, \lambda)} \text{ divides } r - i, \text{ where } i = \text{index}(\lambda)\}$.

Lemma 5.1. *For a field k of characteristic p and the cell algebra $S_R(\mathcal{T}_r, k)$, $\Lambda_p \subseteq \Lambda_0$.*

Proof. Take any $\lambda \in \Lambda_p$ of index i . Let m be the lowest nonzero row of λ , that is, assume $\lambda_m > 0$, $\lambda_j = 0$ for $j > m$. Put $k = k(p, \lambda)$. Let a be the lowest row (i.e., largest integer) such that p^{k+1} does not divide λ_a . Since $\lambda \in \Lambda_p$, p^k divides $r - i$, so $q = (r - i)/p^k$ is an integer. Define a composition μ of r by splitting off the last p^k elements of row a of λ and then adding q additional rows of size p^k :

$$\mu_j = \begin{cases} \lambda_j & \text{if } j < a \\ \lambda_a - p^k & \text{if } j = a \\ p^k & \text{if } j = a + 1 \\ \lambda_{a+l} & \text{if } j = a + l + 1 \text{ for } l = 1, 2, \dots, m - a \\ p^k & \text{if } j = m + 1 + l \text{ for } l = 1, 2, \dots, q \end{cases}$$

Let $C = \{1, 2, \dots, i\}$. Then a composition of i is given by $\mu(C)_j = \mu_j$, $1 \leq j \leq m + 1$. Let S be the semistandard λ -tableau of type $\mu(C)$ where

$$S_{j,l} = \begin{cases} j & \text{for } 1 \leq l \leq \lambda_j, j < a \\ a & \text{for } 1 \leq l \leq \lambda_a - p^k, j = a \\ a + 1 & \text{for } \lambda_a - p^k < l \leq \lambda_a, j = a \\ j + 1 & \text{for } 1 \leq l \leq \lambda_j, a < j \leq m. \end{cases}$$

There is only one standard λ -tableau of type S , namely $s = id$, the identity in \mathfrak{S}_i . The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \bar{i} \rightarrow \bar{r}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \dots, i$.

Now define $D \in D(i, \tau_r, r)$ as follows: D contains $i - p^k$ single elements sets, one set $\{l\}$ for each entry l in rows 1 through a or rows $a + 2$ through $m + 1$ of μ . D also contains p^k sets with $q + 1$ entries: for $1 \leq j \leq p^k$, the j set D_j contains the j th entry in row $a + 1$ and in each of the last q rows of μ . As a composition of i , $\mu(D) = \mu(C)$, so we can take $T = S$ as a semistandard λ -tableau of type $\mu(D)$. Then again there is only one standard λ -tableau of type T , namely $t = id$, the identity in \mathfrak{S}_i .

To define the orbit space $O(D, \mu)$, let $\mathfrak{S}_{\mu_{m+1+j}} \subseteq \mathfrak{S}_\mu$ be the group of permutations of row $m + 1 + j$ of μ . For each $1 \leq j \leq q$, $\mathfrak{S}_{\mu_{m+1+j}} \cong \mathfrak{S}_{p^k}$. Let $G = \prod_{j=1}^q \mathfrak{S}_{\mu_{m+1+j}} \subseteq \mathfrak{S}_\mu$ and for $\sigma \in G$ let $D_\sigma = D\sigma \in D(i, \tau_r, r)$. Then $O(D, \mu) = \{D_\sigma : \sigma \in G\}$. Then $\#O(D, \mu) = o(G) = o((\mathfrak{S}_{p^k})^q) = (p^k!)^q$.

Notice that with our choice of an ordering of the subsets of \bar{r} , we get $\psi_{D_\sigma}(j) = j$, $1 \leq j \leq i$, for any $\sigma \in G$. Then $\psi_{D_\sigma} \circ \phi_C = id$, the identity mapping $\bar{i} \rightarrow \bar{i}$. We have ${}_S C_T^\lambda = {}_S C_i^\lambda = id \cdot r_\lambda \cdot id = r_\lambda$, so

$${}_S C_T^\lambda \circ \psi_{D_\sigma} \circ \phi_C \circ {}_S C_T^\lambda = r_\lambda \cdot id \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) r_\lambda.$$

Then writing $b = {}_{O(C, \mu), S} C_{O(D, \mu), T}^\lambda = \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{\sigma \in G} \psi_{D_\sigma}$, compute

$$\begin{aligned} b \cdot b &= \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{\sigma \in G} \psi_{D_\sigma} \cdot \phi_C \cdot {}_S C_T^\lambda \cdot \sum_{\sigma \in G} \psi_{D_\sigma} \\ &= o(G) \cdot \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \sum_{\sigma \in G} \psi_{D_\sigma} \\ &= o(G) \cdot o(\mathfrak{S}_\lambda) \cdot b. \end{aligned}$$

Then

$$\begin{aligned} b *_R b &= o(G) \cdot o(\mathfrak{S}_\lambda) \cdot \frac{n_R(b)}{n_R(b) n_R(b)} \cdot b \\ &= o(G) \cdot o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_R(b)} \cdot b \\ &= \frac{o(G) o(\mathfrak{S}_\lambda)}{\#O(D, \mu) o(\mathfrak{S}_{\mu(D)})} \cdot b \end{aligned}$$

where $n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(\mathfrak{S}_{\mu(D)})$. Since $\#O(D, \mu) = o(G)$

and $\frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_{\mu(D)})} = \frac{\lambda_a!}{(\lambda_a - p^k)! p^k!} = \binom{\lambda_a}{p^k}$, we get $b *_R b = \binom{\lambda_a}{p^k} \cdot b$.

Computing the bracket gives $\langle C_{O(D, \mu), T}^\lambda, {}_{O(C, \mu), S} C^\lambda \rangle = \binom{\lambda_a}{p^k}$. Since p^k divides λ_a but p^{k+1} does not, it is easily checked that $\binom{\lambda_a}{p^k}$ is not congruent to 0 mod p . So the bracket is not identically zero in $S_R(\tau_r, k)$ and $\lambda \in \Lambda_0$ as desired. \square

We claim that in fact $\Lambda_p = \Lambda_0$, that is, that every irreducible representation of $S_R(\mathcal{T}_r, k)$ corresponds to some $\lambda \in \Lambda_p$. In [2] or [5], a parameterization

of the isomorphism classes of irreducible representations of $S_R(\mathcal{T}_r, k)$ is given. There is one isomorphism class corresponding to the following set of data: i) a set of nonnegative integers s_m, s_{m+1}, \dots, s_M with $s_m > 0, m \geq 0$ such that $r = s_m p^m + s_{m+1} p^{m+1} + \dots + s_M p^M$, ii) a p -restricted partition of s_i for each $m < i \leq M$, and iii) a p -restricted partition of i for some integer $1 \leq i \leq s_m$. (The index of the corresponding irreducible is $r - (s_m - i)p^m$.) We will show that each such set of data corresponds with a unique element $\lambda \in \Lambda_p$, so that the number of isomorphism classes is less than or equal to $\#\Lambda_p$. But we know that the number of isomorphism classes is $\#\Lambda_0$ and by the lemma $\Lambda_p \subseteq \Lambda_0$. So we must have $\Lambda_p = \Lambda_0$.

We will need a certain “decomposition” operation on partitions. For any integer $n > 0$, let λ be a partition of n with R non-zero parts, $\lambda_1 + \lambda_2 + \dots + \lambda_R = n$. For $1 \leq i \leq R$ define the row length differences $\Delta_i = \lambda_i - \lambda_{i+1}$. Then define an integer $k(\lambda) \geq 0$ to be the highest power of p which is less than or equal to at least one Δ_i . Then we can find nonnegative integers q_i, r_i such that $\Delta_i = q_i p^{k(\lambda)} + r_i$ where each $r_i < p^{k(\lambda)}$, each $q_i < p$, and at least one $q_i > 0$. Define $s(\lambda) = \sum_{i=1}^R i \cdot q_i$. We will construct a p -restricted partition of

$s(\lambda)$ and a partition $\bar{\lambda}$ of $n - s(\lambda) p^{k(\lambda)}$ with $k(\bar{\lambda}) < k(\lambda)$. Notice that there are $\Delta_i = q_i p^{k(\lambda)} + r_i$ columns of height i in λ . We break λ into two partitions λ_q, λ_r by placing $q_i p^{k(\lambda)}$ columns of height i in the first partition and r_i columns of height i in the second. Then λ_q is a partition of $i \cdot q_i p^{k(\lambda)} = s(\lambda) p^{k(\lambda)}$ with row differences $q_i p^{k(\lambda)}$. By replacing each set of $p^{k(\lambda)}$ consecutive boxes in a row of λ_q by a single box, we obtain a partition of $s(\lambda)$ with row differences $q_i < p$, i.e., a p -restricted partition of $s(\lambda)$. The second partition λ_r is a partition of $n - s(\lambda) p^{k(\lambda)}$ with row differences $r_i < p^{k(\lambda)}$. Then $k(\lambda_r) < k(\lambda)$ and we take $\bar{\lambda} = \lambda_r$.

We can now replace λ with λ_r and iterate the construction until we reach a case when all r_i are 0. The result is a sequence of nonnegative integers s_m, s_{m+1}, \dots, s_M with $s_m > 0, m \geq 0$ such that $n = s_m p^m + s_{m+1} p^{m+1} + \dots + s_M p^M$ and a p -restricted partition of s_i for each $m \leq i \leq M$. By replacing each box in the partition of s_i by a row of p^i boxes and then joining the resulting partitions (taking the union of the boxes in each row of each partition) we recover uniquely the original partition λ of n . Notice that $k(p, \lambda) = m$.

Now take any isomorphism class of irreducible $S_R(\tau_r, k)$ modules and consider the unique corresponding data i) nonnegative integers s_m, s_{m+1}, \dots, s_M with $s_m > 0$ such that $r = s_m p^m + s_{m+1} p^{m+1} + \dots + s_M p^M$, ii) a p -restricted partition of s_i for each $m < i \leq M$, and iii) a p -restricted partition of s'_m for some integer $1 \leq s'_m \leq s_m$.

Then $s'_m p^m + s_{m+1} p^{m+1} + \dots + s_M p^M = r - (s_m - s'_m) p^m$, so our construction gives a unique partition λ of $r - (s_m - s'_m) p^m$ with $k(p, \lambda) = m$. Then $p^{k(p, \lambda)} = p^m$ divides $r - \text{index}(\lambda) = r - (r - (s_m - s'_m) p^m) = (s_m - s'_m) p^m$, so $\lambda \in \Lambda_p$. So the number of isomorphism classes is $\leq \#\Lambda_p$ as desired.

As remarked above, this proves the following result.

Theorem 5.2. *If k is a field of characteristic p , then $\Lambda_p = \Lambda_0$ in $S_R(\mathcal{T}_r, k)$.*

Corollary 5.1. *If k is a field of characteristic p and $r = ap^l$ for $1 \leq a < p$ and some $l = 0, 1, 2, \dots$, then $S_R(\mathcal{T}_r, k)$ is quasi-hereditary.*

Proof. By corollary 2.1, we must show that $\Lambda = \Lambda_0$, that is, that any $\lambda \in \Lambda$ is actually in $\Lambda_p = \Lambda_0$. So suppose λ is a partition of i for some $0 < i \leq r$. Put $k = k(p, \lambda)$. Since p^k divides λ_j for every j , p^k divides $i = \sum_j \lambda_j$, say $i = bp^k$

for some $b > 0$. Now $bp^k = i \leq r = ap^l < p^{l+1}$ (since $a < p$), so $k \leq l$. Then $r - i = ap^l - bp^k = (ap^{l-k} - b)p^k$, so p^k divides $r - i$ and $\lambda \in \Lambda_p$ as desired. \square

Now consider $S_L(\mathcal{T}_r, k)$ for characteristic p . Define

$$\Lambda_{L,p} = \{\lambda \in \Lambda : p \text{ does not divide } \lambda_j \text{ for at least one } j\} \cup \Lambda(r).$$

Lemma 5.2. *For a field k of characteristic p and the cell algebra $S_L(\mathcal{T}_r, k)$, $\Lambda_{L,p} \subseteq \Lambda_0$.*

Proof. First suppose $\lambda \in \Lambda(r)$, a partition of maximal index r . Then take $\mu = \lambda$ as a composition of r and let $C = \{1, \dots, r\}$. Then $\mu(C) = \mu = \lambda$ and a semi-standard λ -tableau S of type $\mu(C)$ is given by $S_{j,l} = j$, $1 \leq l \leq j$. There is only one standard λ -tableau of type S , namely $s = id$, the identity in \mathfrak{S}_i . The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \bar{r} \rightarrow \bar{r}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \dots, r$.

Define $D \in D(r, \tau_r, r)$ by $D = \{\{1\}, \{2\}, \dots, \{r\}\}$. As a composition of r , $\mu(D) = \mu(C) = \lambda$, so we can take $T = S$ as a semistandard λ -tableau of type $\mu(D)$. Then again there is only one standard λ -tableau of type T , namely $t = id$, the identity in \mathfrak{S}_r .

We have $O(D, \mu) = \{D\}$, $\#O(D, \mu) = 1$, $\psi_D(j) = j$, $1 \leq j \leq r$. Then $\psi_D \circ \phi_C = id : \bar{r} \rightarrow \bar{r}$. We have ${}_S C_T^\lambda = {}_S C_t^\lambda = id \cdot r_\lambda \cdot id = r_\lambda$, so ${}_S C_T^\lambda \circ \psi_D \circ \phi_C \circ {}_S C_T^\lambda = r_\lambda \cdot id \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) r_\lambda$. Then writing $b = {}_{O(C, \mu), S} C_{O(D, \mu), T}^\lambda = \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D$, compute $b \cdot b = \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D \cdot \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D = \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \psi_D = o(\mathfrak{S}_\lambda) \cdot b$. Then $b *_L b = o(\mathfrak{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b)n_L(b)} \cdot b = o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_L(b)} \cdot b = \frac{o(\mathfrak{S}_\lambda)}{\#O(C, \mu)o(\mathfrak{S}_{\mu(C)})} \cdot b$ where $n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\mathfrak{S}_{\mu(C)})$. Since $\#O(C, \mu) = 1$ and $o(\mathfrak{S}_{\mu(C)}) = o(\mathfrak{S}_\lambda)$, we get $b *_L b = b$. Computing the bracket gives $\langle {}_S C_{O(D, \mu), T}^\lambda, {}_{O(C, \mu), S} C^\lambda \rangle = 1 \neq 0$. So the bracket is not identically zero in $S_L(\tau_r, k)$ and $\lambda \in \Lambda_0$ as desired.

Now take any $\lambda \in \Lambda_{L,p}$ of index $i < r$. Let m be the lowest nonzero row of λ , that is, assume $\lambda_m > 0$, $\lambda_j = 0$ for $j > m$. Let a be the largest integer such that p does not divide λ_a . Define a composition μ of r by splitting off the last element of row a of λ and also adding $r - i$ additional rows of length 1:

$$\mu_j = \begin{cases} \lambda_j & \text{if } j < a \\ \lambda_a - 1 & \text{if } j = a \\ 1 & \text{if } j = a + 1 \\ \lambda_{j-1} & \text{if } a + 2 \leq j \leq m + 1 \\ 1 & \text{if } m + 2 \leq j \leq (m + 1) + (r - i) \end{cases}.$$

Let $C = \{1, 2, \dots, i\}$. Then a composition of i is given by $\mu(C)_j = \mu_j$, $1 \leq j \leq m+1$. Let S be the semistandard λ -tableau of type $\mu(C)$ where

$$S_{j,l} = \begin{cases} j & \text{for } 1 \leq l \leq \lambda_j, j < a \\ a & \text{for } 1 \leq l \leq \lambda_a - 1, j = a \\ a+1 & \text{for } l = \lambda_a, j = a \\ j+1 & \text{for } 1 \leq l \leq \lambda_j, a < j \leq m. \end{cases}$$

There is only one standard λ -tableau of type S , namely $s = id$, the identity in \mathfrak{S}_i . The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \bar{i} \rightarrow \bar{r}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \dots, i$.

Now define $D \in D(i, \tau_r, r)$ as follows: Let x be the last element in row λ_a , that is, $x = \lambda_1 + \lambda_2 + \dots + \lambda_a$. D contains $i-1$ single elements sets, one set $\{l\}$ for every $1 \leq l \leq i$ except $l = x$. D also contains one set with $r-i+1$ elements: $\{x, i+1, i+2, \dots, r\}$. As a composition of i , $\mu(D) = \mu(C)$, so we can take $T = S$ as a semistandard λ -tableau of type $\mu(D)$. Then again there is only one standard λ -tableau of type T , namely $t = id$, the identity in \mathfrak{S}_i .

$$\text{We have } O(D, \mu) = \{D\}, \#O(D, \mu) = 1, \psi_D(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ x & \text{if } j > i \end{cases}.$$

Then $\psi_D \circ \phi_C = id : \bar{i} \rightarrow \bar{i}$. We have $sC_T^\lambda = sC_i^\lambda = id \cdot r_\lambda \cdot id = r_\lambda$, so $sC_T^\lambda \circ \psi_D \circ \phi_C \circ sC_T^\lambda = r_\lambda \cdot id \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) r_\lambda$. Then writing $b =_{O(C, \mu), S} C_{O(D, \mu), T}^\lambda = \phi_C \cdot sC_T^\lambda \cdot \psi_D$, compute

$$\begin{aligned} b \cdot b &= \phi_C \cdot sC_T^\lambda \cdot \psi_D \cdot \phi_C \cdot sC_T^\lambda \cdot \psi_D \\ &= \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \psi_D \\ &= o(\mathfrak{S}_\lambda) \cdot b. \end{aligned}$$

Then

$$b *_L b = o(\mathfrak{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b) n_L(b)} \cdot b = o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_L(b)} \cdot b = \frac{o(\mathfrak{S}_\lambda)}{\#O(C, \mu) o(\mathfrak{S}_{\mu(C)})} \cdot b$$

where $n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\mathfrak{S}_{\mu(C)})$. Since $\#O(C, \mu) = 1$ and $\frac{o(\mathfrak{S}_\lambda)}{o(\mathfrak{S}_{\mu(C)})} = \frac{\lambda_a!}{(\lambda_a-1)!1!} = \lambda_a$, we get $b *_L b = \lambda_a \cdot b$.

Computing the bracket gives $\langle C_{O(D, \mu), T}^\lambda, o(C, \mu), sC^\lambda \rangle = \lambda_a$. Since p does not divide λ_a (by definition), $\lambda_a \neq 0$ in k . So the bracket is not identically zero in $S_L(\tau_r, k)$ and $\lambda \in \Lambda_0$ as desired. \square

We claim that $\Lambda_{L,p} = \Lambda_0$, that is, that every irreducible representation of $S_L(\mathcal{T}_r, k)$ corresponds to some $\lambda \in \Lambda_{L,p}$. [5] gives the following parameterization of the isomorphism classes of irreducible representations of $S_L(\mathcal{T}_r, k)$. There is one isomorphism class corresponding to the following set of data: i) a set of nonnegative integers s_m, s_{m+1}, \dots, s_M with $m \geq 0, s_m > 0$ such that $r = s_m p^m + s_{m+1} p^{m+1} + \dots + s_M p^M$, ii) a p -restricted partition of s_i for each

$m < i \leq M$, iii) a p -restricted partition of s_m if $m > 0$ or a p -restricted partition of i for some integer $1 \leq i \leq s_0$ if $m = 0$. (The index of the corresponding irreducible is $r - (s_0 - i)$.) We will show that each such set of data corresponds with a unique element $\lambda \in \Lambda_{L,p}$, so that the number of isomorphism classes is less than or equal to $\#\Lambda_{L,p}$. We know that the number of isomorphism classes is $\#\Lambda_0$ and by the lemma $\Lambda_{L,p} \subseteq \Lambda_0$, so we must have $\Lambda_{L,p} = \Lambda_0$.

Consider first a set of data for the case $m > 0$. Then we have $r = s_m p^m + s_{m+1} p^{m+1} + \cdots + s_M p^M$ and a p -restricted partition of s_m for all m . So by the construction preceding theorem 5.2 there is a unique partition λ of index r corresponding to the data, and since the index of λ is r we have $\lambda \in \Lambda_{L,p}$. Next consider a set of data for the case $m = 0$. Putting $s'_0 = i$ we have $s'_0 + s_1 p^1 + \cdots + s_M p^M = r - (s_0 - s'_0)$ with p -restricted partitions of $s'_0 = i$ and all $s_i, i > 0$. The result is a unique partition λ of $r - (s_0 - i)$ with $k(p, \lambda) = m = 0$. But $k(p, \lambda) = 0$ means that at least one row length λ_j of λ is not divisible by $p^{k(p, \lambda)+1} = p$, that is, that $\lambda \in \Lambda_{L,p}$. So corresponding to each set of data there is a unique element $\lambda \in \Lambda_{L,p}$ as desired. As remarked above, this proves the following theorem.

Theorem 5.3. *If k is a field of characteristic p , then $\Lambda_{L,p} = \Lambda_0$ in $S_L(\mathcal{T}_r, k)$.*

Notice that if $p \geq r$ then $\Lambda_0 = \Lambda_{L,p} = \Lambda$ and $S_L(\mathcal{T}_r, k)$ is quasi-hereditary. However, when $r > p$ we have $\Lambda_0 = \Lambda_{L,p} \neq \Lambda$ and $S_L(\mathcal{T}_r, k)$ is not quasi-hereditary for the given poset structure Λ .

Now consider the case when M contains the rook monoid \mathfrak{R}_r .

Theorem 5.4. *Assume M contains the rook monoid \mathfrak{R}_r . Then for any field k , $\Lambda_0 = \Lambda$ for both cell algebras $S_L(M, k)$ and $S_R(M, k)$. Both $S_L(M, k)$ and $S_R(M, k)$ are quasi-hereditary.*

Remark: If M is just the rook monoid, $M = \mathfrak{R}_r$, then $S_L(M, k)$ and $S_R(M, k)$ are both actually cellular algebras and are anti-isomorphic as algebras.

Proof. Take any partition λ of i , $0 \leq i \leq r$. We must show that $\lambda \in \Lambda_0$. Let $m \geq 0$ be the smallest integer such that $\lambda_j = 0$ for $j > m$. Define a composition μ of r by adding $r - i$ rows of length 1 to λ . So

$$\mu_j = \begin{cases} \lambda_j & \text{if } j \leq m \\ 1 & \text{if } m+1 \leq j \leq m+r-i \\ 0 & \text{if } j > m+r-i. \end{cases}$$

Let $C = \{1, 2, \dots, i\}$. Then $\mu(C) = \lambda$. Let S be the semistandard λ -tableau of type $\mu(C)$ where $S_{j,l} = j$, $1 \leq l \leq \lambda_j$, $1 \leq j \leq m$. There is only one standard λ -tableau of type S , namely $s = id$, the identity in \mathfrak{S}_i . The orbit $O(C, \mu) = \{C\}$, so $\#O(C, \mu) = 1$. Also, $\phi_C : \bar{i} \rightarrow \bar{r}$ is the identity $\phi_C(j) = j$, $j = 1, 2, \dots, i$.

Define $D \in D(i, r)$ by $D = \{\{j\} : 1 \leq j \leq i\}$. M contains the rook monoid \mathfrak{R}_r , so it contains the map $\alpha : \bar{r} \cup 0 \rightarrow \bar{r} \cup 0$ given by $\alpha(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ 0 & \text{if } j > i \end{cases}$.

Then $D = D_\alpha \in D(i, M, r)$. As a composition of i , $\mu(D) = \mu(C) = \lambda$, so we can take $T = S$ as a semistandard λ -tableau of type $\mu(D)$. Then again there is only one standard λ -tableau of type T , namely $t = id$, the identity in \mathfrak{S}_i .

We have $O(D, \mu) = \{D\}$, $\#O(D, \mu) = 1$, $\psi_D(j) = \begin{cases} j & \text{if } 1 \leq j \leq i \\ 0 & \text{if } j > i \end{cases}$. Then

$\psi_D \circ \phi_C = id : \bar{i} \rightarrow \bar{i}$.

We have ${}_S C_T^\lambda = {}_S C_t^\lambda = id \cdot r_\lambda \cdot id = r_\lambda$, so ${}_S C_T^\lambda \circ \psi_D \circ \phi_C \circ {}_S C_T^\lambda = r_\lambda \cdot id \cdot r_\lambda = r_\lambda \cdot r_\lambda = o(\mathfrak{S}_\lambda) r_\lambda$. Then writing $b = {}_{O(C, \mu), S} C_{O(D, \mu), T}^\lambda = \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D$, compute $b \cdot b = \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D \cdot \phi_C \cdot {}_S C_T^\lambda \cdot \psi_D = \phi_C \cdot o(\mathfrak{S}_\lambda) \cdot r_\lambda \cdot \psi_D = o(\mathfrak{S}_\lambda) \cdot b$.

Then $b *_L b = o(\mathfrak{S}_\lambda) \cdot \frac{n_L(b)}{n_L(b)n_L(b)} \cdot b = o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_L(b)} \cdot b = \frac{o(\mathfrak{S}_\lambda)}{\#O(C, \mu)o(\mathfrak{S}_{\mu(C)})} \cdot b$ where $n_L(b) = n_L(O(C, \mu)) = \#O(C, \mu) \cdot o(\mathfrak{S}_{\mu(C)})$. Since $\#O(C, \mu) = 1$ and $\mathfrak{S}_\lambda = \mathfrak{S}_{\mu(C)}$, we get $b *_L b = b$.

Computing the bracket in $S_L(M, k)$ gives $\langle C_{O(D, \mu), T}^\lambda, {}_{O(C, \mu), S} C^\lambda \rangle = 1 \neq 0$. So the bracket is not identically zero in $S_L(M, k)$ and $\lambda \in \Lambda_0$ as desired.

Similarly, $b *_R b = o(\mathfrak{S}_\lambda) \cdot \frac{n_R(b)}{n_R(b)n_R(b)} \cdot b = o(\mathfrak{S}_\lambda) \cdot \frac{1}{n_R(b)} \cdot b = \frac{o(\mathfrak{S}_\lambda)}{\#O(D, \mu)o(\mathfrak{S}_{\mu(D)})} \cdot b$ where $n_R(b) = n_R(O(D, \mu)) = \#O(D, \mu) \cdot o(\mathfrak{S}_{\mu(D)})$. Since $\#O(D, \mu) = 1$ and $\mathfrak{S}_\lambda = \mathfrak{S}_{\mu(D)}$, we get $b *_R b = b$.

Computing the bracket in $S_R(M, k)$ again gives $\langle C_{O(D, \mu), T}^\lambda, {}_{O(C, \mu), S} C^\lambda \rangle = 1 \neq 0$. So the bracket is not identically zero in $S_R(M, k)$ and $\lambda \in \Lambda_0$ as desired.

By corollary 2.1, cell algebras with $\Lambda_0 = \Lambda$ are quasi-hereditary. \square

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